

# ADVANCED METHODS IN APPLIED MATHEMATICS

by

Richard Courant

Notes by

Charles DePrima and John R. Knudsen

Summer 1941

New York University

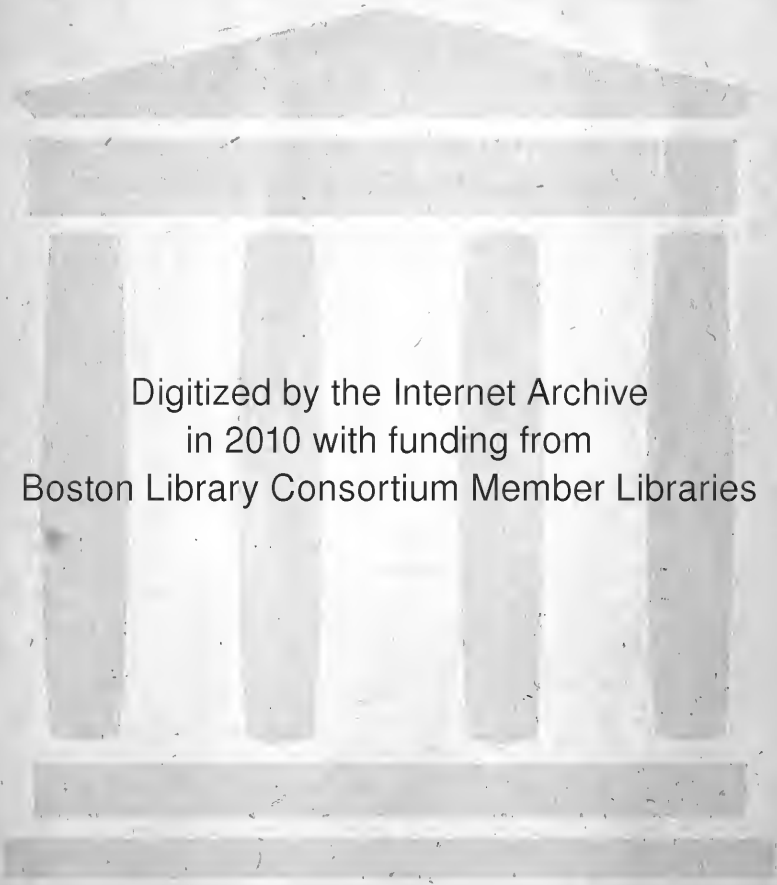
Institute for Mathematics and Mechanics

Boston University



College of Liberal Arts  
Library

Graduate School



Digitized by the Internet Archive  
in 2010 with funding from  
Boston Library Consortium Member Libraries

<http://www.archive.org/details/advancedmethods00cour>





## ADVANCED METHODS IN APPLIED MATHEMATICS

Lecture Course by Professor Richard Courant under the auspices of the Engineering Defense Training Program of the United States Office of Education at the College of Engineering, New York University. during the Summer of 1941.

### References

- J. P. Den Hartog: Mechanical Vibrations
- T. von Karman and M. Biot: Mathematical Methods  
in Engineering
- S. Timoshenko: Vibration Problems in Engineering
- Lord Rayleigh: Theory of Sound, (2 Vols.)
- R. Courant: Differential and Integral Calculus

GRADUATE  
JANUARY 7, 1954  
G1091  
SCHEID

# T A B L E   O F   C O N T E N T S

935  
P41

INTRODUCTION .....	1
--------------------	---

## C H A P T E R   O N E

### SYSTEMS WITH A FINITE NUMBER OF DEGREES OF FREEDOM

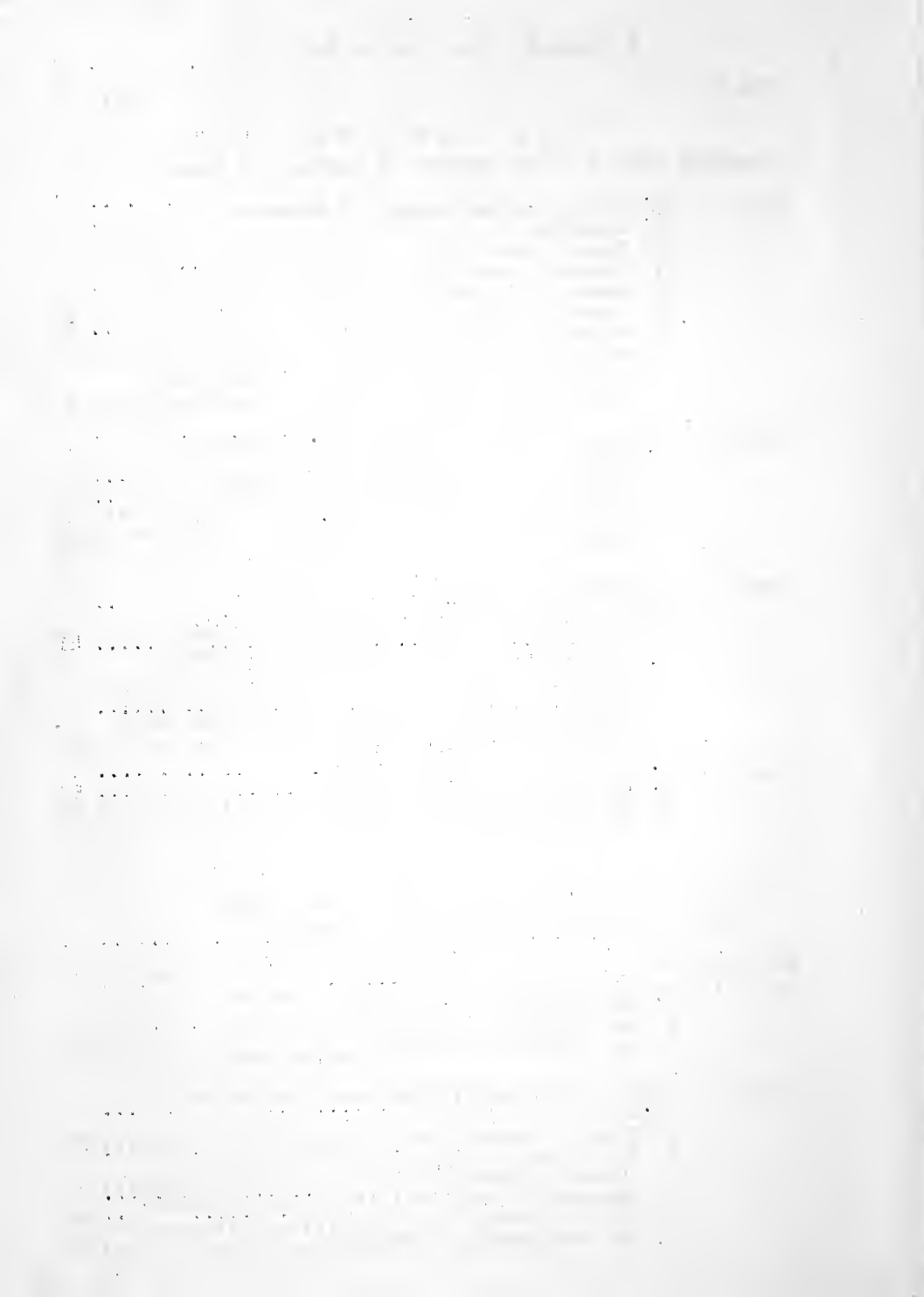
Part I: Systems with One Degree of Freedom	
1. Examples .....	1
2. Free Vibrations .....	3
3. Forced Vibrations .....	7
4. General External Force $f(t)$ .....	9
5. Energy Balance .....	11
6. Resonance Phenomena .....	12
7. Recording Instruments .....	15
8. Superposition of Different Vibrations with Random Phases .....	18
Part II: Systems of Several Degrees of Freedom	
1. Two Degrees of Freedom .....	26
2. Systems with $n$ Degrees of Freedom .....	30
3. Influence of External Forces .....	36
4. Influence of Friction .....	36
5. Energy Balance .....	38
Part III: Extremum Properties of the Natural Frequencies and Modes	
1. The Frequencies as Successive Minima .....	38
2. Constraints - Frequencies as Maximum - Minimum .....	41
3. Loss of Degrees of Freedom by Continuous Processes. Behavior of the Spectrum under Changes of the System .....	46
Part IV: The Method of Perturbations	
1. An Example of the Method .....	48
2. The Method in General .....	53

## C H A P T E R   T W O

### VIBRATION AND EQUILIBRIUM PROBLEMS OF SYSTEMS

#### WITH INFINITELY MANY DEGREES OF FREEDOM

Introduction .....	56
Part I: Equations of the String and Membrane	
1. The String .....	58
2. The Membrane .....	62
3. The Question of Small Deviations .....	65
Part II: Solution of Equilibrium Problems for the Membrane	
1. General Formulation .....	66
2. The Boundary Value Problem for the Circular Membrane .....	67
3. Integral Representation of the Solution for the Circle .....	72
4. The Rectangular Membrane .....	74



Part III:	Method of Finite Differences	
	Introduction .....	79
	1. The Laplace Difference Equation .....	79
	2. Boundary Value Problem in a Net .....	82
	3. Theoretical Remarks .....	88
Part IV:	Non-homogeneous Equilibrium Equation. Green's Function.	
	1. The Non-homogeneous and Homogeneous Equations .....	91
	2. The Boundary Value Problem for the Poisson Equation in a Square Domain .....	94
	3. Equilibrium under Concentrated External Forces .....	97
	4. Green's Function for the Elastic String ..	100
	5. Green's Function for the Elastic Membrane .....	102
	6. Further Examples of Green's Function .....	105
	7. Realization of Green's Function by Physical Means .....	109
Part V:	The Vibrating String and Membrane	
	1. General Theory of Natural Vibrations and Natural Modes .....	111
	2. The Vibrating String .....	115
	3. The Impulse Method .....	120
	4. The Non-homogeneous String .....	124
	5. Strings with Other Boundary Conditions ...	126
	6. The Square Membrane .....	128
	7. The Circular Membrane .....	132
Part VI:	A Summary of the General Theory of Eigen-values	
	Introduction .....	135
	1. Orthogonality and Completeness of the Eigen-functions .....	137
	2. Equivalence of the Impulse and Expansion Methods .....	143
	3. Green's Function Expanded in Terms of Eigen-functions .....	146
	4. Asymptotic Behavior of the Eigen-value ...	148

## C H A P T E R   T H R E E

### VARIATIONAL METHODS

Part I:	Introduction .....	152
Part II:	Mathematical Preparation	
	1. Homogeneous Quadratic "Functionals" .....	157
	2. Technicalities of the Calculus of Variations .....	161
	3. Natural Boundary Conditions .....	167
	4. Remarks Concerning Approximation of Functions .....	175
Part III:	The Rayleigh-Ritz Method	
	1. Equilibrium Problems .....	177
	2. Further Examples and Remarks .....	179
	3. Applications to the Theory of Vibrations .	182
Part IV:	Critical Remarks. Other Methods .....	187

.....  
.....  
.....  
.....

.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

.....  
.....

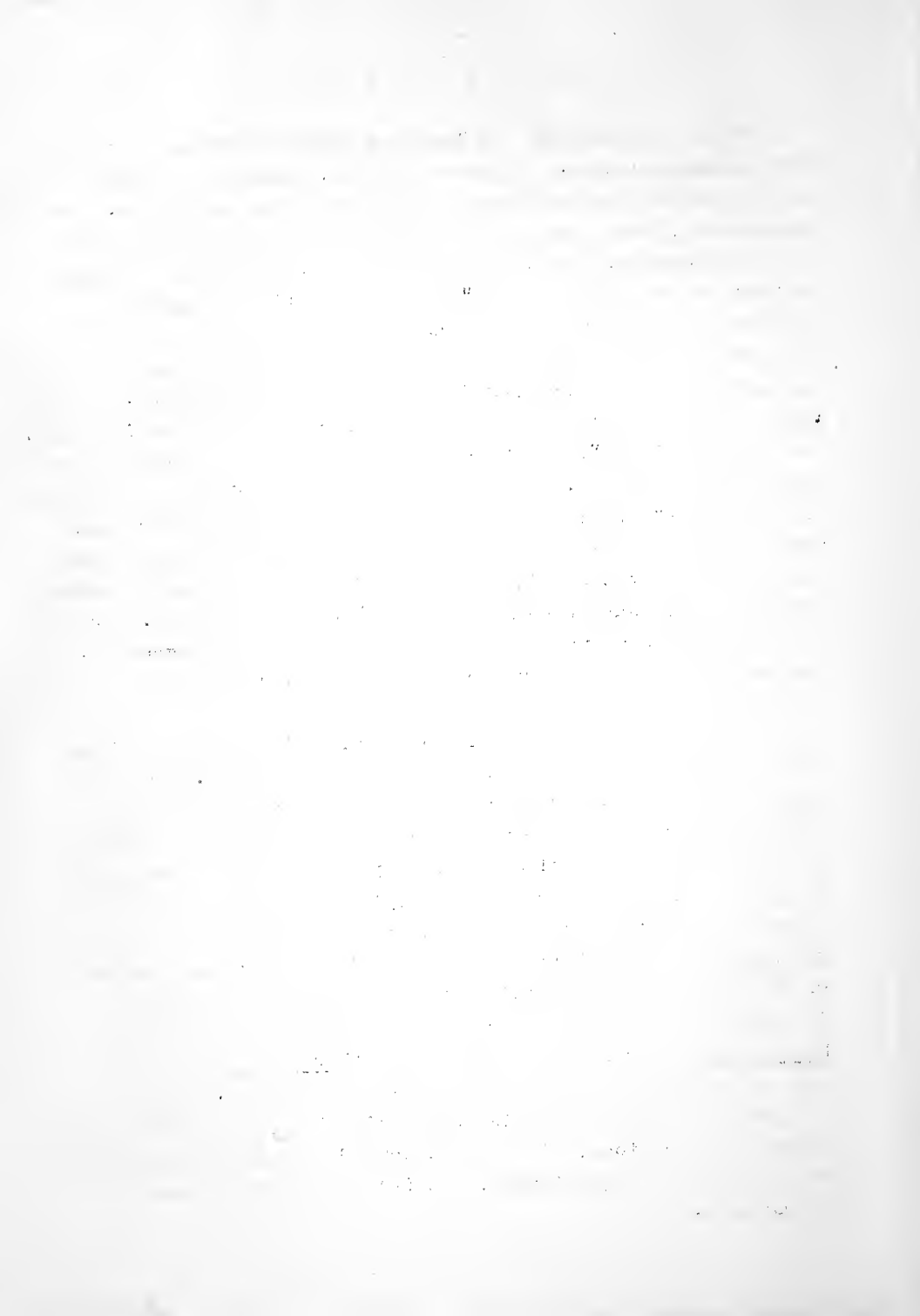
.....  
.....

## I N T R O D U C T I O N

"Pure mathematics" and "applied mathematics" essentially form an organic unit. In spite of this, however, a narrow specialization has developed in each of these two fields. In consequence, there exists an unfortunate tendency on the part of the representatives of one field to disregard the endeavors of those of the other. The "pure" mathematician sometimes considers a difficult problem as solved if he is able to demonstrate that a logical contradiction follows from the assumption that the problem does not possess a solution. Difficult as such a logical achievement may be in certain cases, the claim to have "solved" the problem in this manner will not impress the engineer. The latter is not so much interested in an "existence" proof as he is in the actual construction and mastering of the explicit solution. On the other hand, those who are interested solely in practical results and applications are not always satisfied merely with "routine recipes". For them a deeper penetration into the mathematical background is indispensable if they would become independent of rigid patterns.

This course will emphasize mathematical methods which are connected with important types of applied problems. The isolation of the essential mathematical features from the physical features of a given problem often exhibits the core of the problem and shows that apparently different and independent phenomena may have identical underlying structures. For example, the mathematical formulation of mechanical vibration problems on the one hand and of electrical vibration problems on the other exposes certain structural similarities between the two fields. The tendency must be to consider such a diagnostic procedure as a basis for a therapeutic treatment, that is, for the actual mastering of the solution.

The main topics of this course will be (1) the mathematical theory of vibrations, with a view toward discussing questions of stability and equilibrium, and (2) the theory of wave propagation.





These physical problems will be formulated mathematically as problems concerning differential equations. Differential equations express the general physical laws that govern an entire class of mechanical and electrical phenomena. A particular phenomenon within the class is characterized by additional conditions such as initial or boundary conditions or, possibly, a combination of both. The differential equation will yield the totality of solutions to the general problem and the added conditions will determine the specific solution desired for the problem under consideration.

The type of differential equation that will arise will depend, among other things, on the number of degrees of freedom of the physical system under investigation. Thus, if the physical system has but one degree of freedom we shall have to deal with one ordinary differential equation; if the system has a finite number, say  $n$ , degrees of freedom we must consider a system of  $n$  ordinary differential equations. Finally, in the case of continuous bodies with infinitely many degrees of freedom we shall encounter partial differential equations.

The simplest type of vibration problem is given by the differential equation

$$(I) \quad m\ddot{u} + r\dot{u} + ku = f(t) \quad .$$

Here  $u = u(t)$  is the unknown function, depending upon the time  $t$ , which is sought; for example,  $u$  might be the displacement of a vibrating particle in a mechanical system. The dot indicates differentiation with respect to time  $t$ ; thus

$$\dot{u} = \frac{du}{dt} \quad , \quad u = \frac{d^2u}{dt^2} \quad .$$

The constants  $m$  (mass),  $r$  (coefficient of friction), and  $k$  (spring constant or elastic restoring force<sup>\*</sup>) characterize the mechanical system. The function  $f(t)$  is a prescribed external (or impressed) force.

Any physical problem concerning a particular phenomenon of the class described by (I) amounts mathematically to the determining of a particular solution  $u = u(t)$  of equation (I) having the property that the two additional conditions

---

\* Here and elsewhere throughout the notes the quantity  $k$  is the elastic restoring force "per unit displacement".



$$u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0$$

are satisfied. The constants  $u_0$  and  $\dot{u}_0$ , the prescribed "displacement" and "velocity", respectively, of the system at time  $t = 0$ , constitute the so-called initial state of the vibration phenomenon.

The simplest continuous vibrating body is an elastic string of linear density  $\rho$ , stretched with equal tension  $S$  between fixed ends, say  $x = 0$  and  $x = \ell$  along the  $x$ -axis. We suppose the string movable in such a way that each point moves perpendicularly to the  $x$ -axis and all points remain in a fixed plane containing the  $x$ -axis. Then if we call  $u$  the deflection of a point of the string,  $u$  will be a function of the position  $x$  of the point and of the time  $t$ :  $u = u(x, t)$ . The motion of the string will be governed by the partial differential equation

$$(II) \quad Su_{xx} = \rho u_{tt}.$$

The subscripts, as always, will denote partial differentiation with respect to that subscript; thus

$$u_x = \frac{\partial u}{\partial x}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad u_{xt} = \frac{\partial^2 u}{\partial x \partial t}.$$

To determine the motion in any specific case we must know the initial deflection  $u(x, 0) = f(x)$  and the initial velocity  $u_t(x, 0) = g(x)$  of the string, and, in addition, we must take care that the boundary conditions  $u(0, t) = u(\ell, t) = 0$  are satisfied for all values of the time  $t$ . Thus, we must solve both an initial value problem and a boundary value problem in connection with equation (II).

To study the motion of an elastic membrane we shall have to discuss similar initial value and boundary value problems. In this case the deflection  $u = u(x, y, t)$  is a function of three variables  $x, y, t$ , and the partial differential equation involved is

$$(III) \quad S\Delta u = \rho u_{tt}.$$



Here  $S$  represents the tension along the boundary of the membrane and  $\rho$  the surface density of the membrane. The differential operator  $\Delta u = u_{xx} + u_{yy}$  is the so-called Laplace operator.

An elastic plate, which is of even greater importance to the engineer, will move in accordance with the partial differential equation of the fourth order

$$(IV) \quad S \Delta \Delta u = \rho u_{tt}.$$

Again the initial state and the additional conditions at the boundary will be instrumental in determining the specific solution desired.

In many important applications elastic bodies cannot be considered as entirely uniform throughout their extent. They may consist of various parts each with different elastic properties. Such discontinuities will be taken care of by imposing subsidiary conditions on the differential equation. The physical phenomenon can then be handled mathematically by means of these "discontinuity" conditions, as we shall see, e.g., in the problem of plates reinforced by beams.

Problems of equilibrium correspond mathematically to the case when, in the differential equation, the function  $u$  does not depend on the time  $t$ . For example, the equilibrium equation corresponding to (III) is

$$(V) \quad \Delta u = 0$$

where  $u = u(x, y)$ . A specific solution is determined by imposing a boundary condition, say a prescribed deflection along the boundary of the membrane. It should be noted that no initial value problem occurs here. Likewise, the equilibrium equation for plates is

$$(VI) \quad \Delta \Delta u = 0.$$

Problems occurring in wave propagation, in particular those concerning light and electro-magnetic waves, are treated by the differential equation (III) written in a slightly different form:



(VII)

$$\Delta u = c^{-2} u_{tt}$$

where  $c$  is the velocity of light. In general,  $u$  is, in this case, a function of three space variables and one time variable, i.e.,  $u = u(x, y, z, t)$  and the Laplace operator is

$\Delta u = u_{xx} + u_{yy} + u_{zz}$ . The essential difference between (III) and (VII) will appear in the character of the additional conditions imposed upon the differential equation.

Many relevant problems of the types mentioned above can be handled with great success by treating them as maximum and minimum problems, or, as we shall say, variational problems. Such variational statements, in most cases, will involve the potential and kinetic energy of the physical system. Many times such a formulation renders a quite difficult problem accessible to numerical treatment (for example, the so-called Rayleigh-Ritz method).

Our goal in this course will be to select and to present certain methods and viewpoints concerning all these questions in a manner which is not so easily accessible in current literature.





# CHAPTER ONE

## SYSTEMS WITH A FINITE NUMBER OF DEGREES OF FREEDOM

### Part I. Systems with One Degree of Freedom

#### 1. Examples

We assume that the reader is to some degree familiar with the simpler problems of vibrating systems of one degree of freedom. Nevertheless, we give a brief summary of the more important points of this topic.

Actually, mechanical systems with one degree of freedom cannot be constructed, but systems with a motion approximating that of one degree of freedom can be made. For example, in figure 1 is depicted a mass particle attached to the center of a horizontal spring fastened at both ends. The particle is free to move in a straight horizontal line. We idealize the system by imagining the weight of the spring to be so small compared with the mass of the particle as to be negligible. The state of motion of the system is then described at all times by one number, for example, the distance of the particle from its equilibrium position. Thus the idealized case, at least, is a system with one degree of freedom.



Fig. 1

Another example is that shown in figure 2. A tautly stretched string of length  $\ell$  fastened at both ends has a mass particle attached to its center. Again we consider the weight of the string to be a negligible quantity. The particle is free to oscillate in a line normal to the equilibrium position of the string. We suppose also, since the system is to have but one degree of freedom, that the

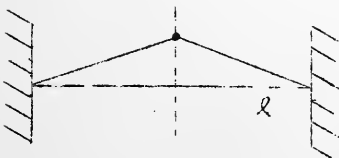


Fig. 2



particle moves always in a fixed plane through the equilibrium position of the string. Concerning the tension in the string, we assume simply that it is proportional to the increase in length.

The "simple" pendulum which makes small oscillations in a fixed plane is another familiar system of one degree of freedom. The value of the angular displacement at any time determines the position of the pendulum arm.

Here, too, the arm of the pendulum is to be considered weightless.

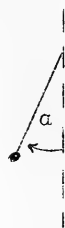


Fig. 3

A study of the above ideal systems is important to the investigation of actual mechanical systems which are approximately ideal. Moreover, many mechanical systems of more than one degree of freedom can be approximated by the ideal systems above. For example, in the treatment of a vibrating microphone diaphragm we can consider the diaphragm as having its total mass concentrated at its center. The system is then closely analogous to the system of figure 2.

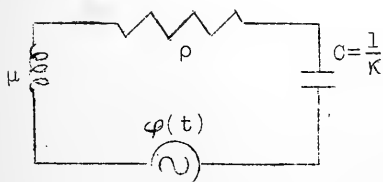


Fig. 4

Vibrating systems do exist which approximate the ideal systems above more closely than do actual mechanical systems, namely, oscillatory electrical circuits.

In figure 4 we have such a circuit containing an inductance  $\mu$ , resistance  $\rho$ , capacitance  $C = 1/K$ , and with an impressed electromotive force  $\varphi(t)$ .

The ideal vibrating system of one degree of freedom is described mathematically by the differential equation

$$(I) \quad m\ddot{u} + r\dot{u} + ku = f(t) \quad .$$

In different physical problems the constants  $m$ ,  $r$ , and  $k$ , and the function  $u(t)$  do not represent the same quantities. Thus, in the first two examples  $m$  represents the mass of the particle,  $r$  the coefficient of friction or damping,  $k$  the elastic



restoring force, and  $u(t)$  the linear displacement of the particle from its equilibrium position. In the third example  $u(t) = \alpha(t)$  where  $\alpha$  is the angular displacement of the pendulum from its vertical position. In the electrical circuit  $m = \mu$ ,  $r = \mathcal{R}$ ,  $k = \mathcal{K}$ , and  $u(t)$  may represent either the voltage  $E$  or the current  $I$  across the condenser. In the former case  $f(t) = -\mathcal{K}\varphi(t)$ ; in the latter  $f(t) = \dot{\varphi}(t)$ .

One might ordinarily have expected to find a significant difference between the mathematical formulation of mechanical vibration problems on the one hand and that of electrical vibration problems on the other. Such a difference we see does not exist. The formulation of both problems is given by equation (I). To accomplish such unifying processes is in agreement with the purpose of this course as stated in the introduction.

## 2. Free Vibrations

We consider first an ideal vibrating system in which no impressed force is present. The equation describing such a system of free vibrations is clearly

$$(1) \quad m\ddot{u} + r\dot{u} + ku = 0,$$

namely, equation (I) with  $f(t)$  set equal to zero.

To find the general solution of (1) we find first two non-trivial (also non-singular) independent particular solutions. An arbitrary linear combination of the two particular solutions will then be the desired general solution, i.e., an expression representing the totality of all solutions. Since (1) is linear and homogeneous we can obtain a particular solution in the form of an exponential function  $e^{\lambda t}$  where  $\lambda$  is a constant to be determined. If we substitute  $e^{\lambda t}$  for  $u$  in (1) and remove the factor  $e^{\lambda t}$  (which is different from zero), we obtain the quadratic equation

$$(2) \quad m\lambda^2 + r\lambda + k = 0$$

for the quantity  $\lambda$ . Denoting the two roots of (2) by  $\lambda_1$  and  $\lambda_2$  we have



$$u = e^{\lambda_1 t} \quad \text{and} \quad u = e^{\lambda_2 t}$$

as two particular solutions of equation (1).

The nature of the general solution and, hence, of the motion it describes, depends upon the nature of the roots. There are three cases to consider.

1)  $r^2 - 4mk > 0$ . In this case  $\lambda_1$  and  $\lambda_2$  are real, unequal and both are negative if we assume, as is natural, that  $m > 0$ ,  $k \geq 0$ . The general solution is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

2)  $r^2 - 4mk = 0$ . Here equation (2) has a double root  $\lambda = -r/2m$ . Thus  $u = e^{-\frac{r}{2m}t}$  is the only solution obtainable from the auxiliary equation (2). A second particular solution, however, can be shown to be given by  $u = te^{-\frac{r}{2m}t}$ . The general solution in this case is

$$u(t) = (c_1 + c_2 t) e^{-\frac{r}{2m}t}.$$

Exercise: Verify directly that  $u = te^{-\frac{r}{2m}t}$  is a solution of (1) if  $r^2 - 4mk = 0$ . Also, derive this solution by considering the expression

$$\frac{e^{\lambda_1 t} - e^{\lambda t}}{\lambda_1 - \lambda} \quad (\lambda_1 \neq \lambda = -\frac{r}{2m})$$

as  $\lambda_1 \rightarrow \lambda$ .

The motion described by these two cases is the so-called aperiodic motion shown in figure 5. As time increases the "displacement" approaches zero asymptotically without oscillating about  $u = 0$ . The effect of damping is so great that it prevents the elastic force from setting up oscillatory motions. In general, we shall not be concerned with aperiodic motions.





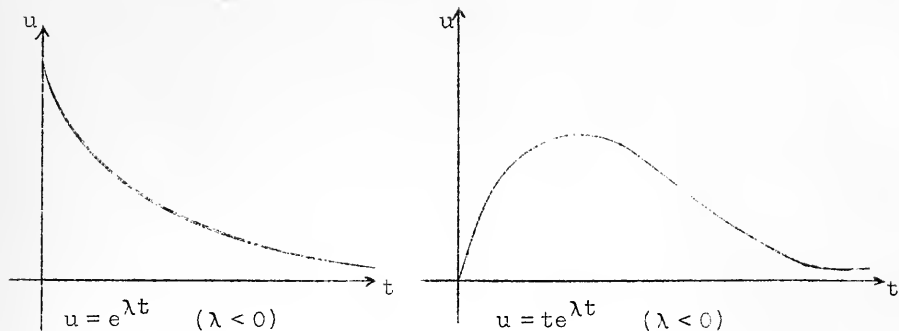


Fig. 5

3)  $r^2 - 4mk < 0$ . In this case  $\lambda_1$  and  $\lambda_2$  are complex conjugates. Setting

$$r^2 - 4mk = -4m^2\nu^2$$

enables us to write the general solution of (1) in the form

$$(3) \quad u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = e^{-\frac{r}{2m}t} (c_1 e^{i\nu t} + c_2 e^{-i\nu t})$$

or, since  $e^{i\nu t} = \cos \nu t + i \sin \nu t$ ,

$$(3') \quad u(t) = e^{-\frac{r}{2m}t} (c_1' \cos \nu t + c_2' \sin \nu t) .$$

If we let  $c_1' = A \cos \nu \delta$  and  $c_2' = A \sin \nu \delta$ , where  $A$  and  $\delta$  are new arbitrary constants, (3') becomes

$$(3'') \quad u(t) = A e^{-\frac{r}{2m}t} \cos \nu(t - \delta) .$$

The constant  $A$  is called the amplitude of the motion (It is the greatest "displacement" attainable) and  $\nu \delta$  the phase. Also, the quantity

$$\nu = \sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}$$

is the frequency of the motion. Before discussing the motion which this solution represents, we introduce a new form for the solution, a form which we shall encounter in the next section, namely,



$$(3''') \quad u(t) = \alpha e^{-\frac{r}{2m}t} e^{i\nu t}$$

where  $\alpha$  is complex:  $\alpha = a + ib$ . The solution in this form with complex "amplitude"  $\alpha$  is very useful in the mathematical treatment.

The motion in this case is quite different from the aperiodic motion of the first two cases. This time the damping is small compared with the elastic force and an oscillatory motion is actually achieved. However, the damping is not so small as to be considered negligible. The motion,

which is one of damped harmonic oscillations, is best described by the solution

$$u(t) = A e^{-\frac{r}{2m}t} \cos \nu(t - \delta).$$

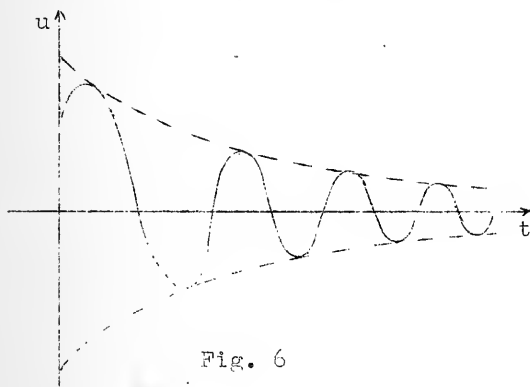


Fig. 6

The oscillations behave like sine waves with a circular frequency

$$\nu = \sqrt{\frac{k}{m} - \frac{r^2}{4m^2}}$$

except that the maximum value of the displacement attained with each oscillation is not constant. Rather, it is given by the

expression  $A e^{-\frac{r}{2m}t}$  and thus decreases exponentially as time  $t$  increases (see figure 6). The rate of decrease of the maximum

displacement depends upon the quantity  $r/2m$ . The greater this quantity is (or, for fixed  $m$ , the greater  $r$  is) the faster will the motion die down. Frequently the quantity  $r/2m$  is called the logarithmic decrement indicating that the logarithm of the maximum displacement decreases at the

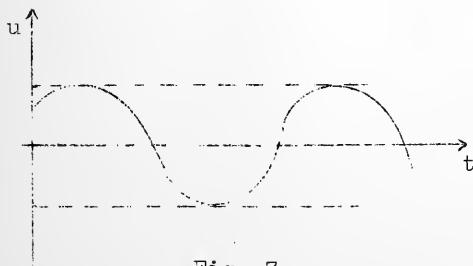


Fig. 7



rate  $2/rm$ . In case no damping is present ( $r=0$ ), we obtain simple harmonic oscillations with natural frequency  $\omega_0 = \sqrt{k/m}$ .

For the differential equation (1), which characterizes an entire class of physical phenomena, we have obtained the general solution. However, we are not so much concerned with a totality of solutions as we are with a particular solution describing some special phenomenon of the class. We are interested mainly in a solution which describes a motion whose initial state is known. Such a particular solution is obtained by using the initial conditions to determine, say, the amplitude  $A$  and the phase  $\delta$  (or, simply,  $A$  and  $\delta$ ). For example, if the initial state is

$$u(0) = 0 \quad \text{and} \quad \dot{u}(0) = \dot{u}_0,$$

then

$$u(t) = \frac{\dot{u}_0}{\omega} e^{-\frac{r}{2m}t} \sin \omega t$$

is the particular solution.

### 3. Forced Vibrations

If an external force  $f(t) \neq 0$  is impressed upon the physical system the resulting motion is one of forced vibrations. Characterizing such a system is the non-homogeneous differential equation

$$(4) \quad mu + r\dot{u} + ku = f(t).$$

We know the general solution of (4) to be the superposition of the general solution of the homogeneous equation (1) and any particular solution of (4). Our problem, then, amounts to finding a single particular solution of (4). We simplify our problem first by assuming the external force to be periodic with the simple form

$$f(t) = ce^{i\omega t}$$

where  $\omega$  is the frequency and  $c$  the complex amplitude. Later we will consider the more general case where we do not make this assumption concerning  $f(t)$ .



Since equation (4) is linear we try to find a solution of the form

$$(5) \quad u = \gamma e^{i\omega t}$$

where  $\gamma$  is a constant to be determined. Equation (5) represents an oscillatory motion which is in rythm with the impressed force since the frequency  $\omega$  is the same as that for  $f(t)$ . If we substitute the above trial value of  $u$  in (4) we get

$$-m\omega^2 \gamma + ir\omega \gamma + k\gamma = c$$

or

$$(6) \quad \gamma = \frac{c}{-m\omega^2 + ir\omega + k}$$

Hence

$$(5') \quad u = \frac{ce^{i\omega t}}{-m\omega^2 + ir\omega + k}$$

is a particular solution of (4).

To simplify the calculations we write (6) in a different form, i.e.,

$$(7) \quad \gamma = \frac{c(k - m\omega^2 - ir\omega)}{(k - m\omega^2)^2 + r^2\omega^2} = c\mu e^{-i\omega\delta}$$

where the positive distortion factor  $\mu$  and the phase displacement  $\omega\delta$  are expressed in terms of  $m$ ,  $r$ ,  $k$ , by the relations

$$(8) \quad \mu^2 = \frac{1}{(k - m\omega^2)^2 + r^2\omega^2}$$

$$(8') \quad \begin{cases} \sin \omega\delta = r\omega\mu \\ \cos \omega\delta = (k - m\omega^2)\mu \end{cases} .$$

Our solution (5) now has the form

$$(9) \quad u = c\mu e^{i\omega(t - \delta)} .$$





The physical meaning here is this. If a force of the form, say,  $c \cos \omega t$  is impressed on the vibrating system, the resulting motion will be given by  $u = c \mu \cos \omega(t - \delta)$ , (and similarly for  $c \sin \omega t$ ). Hence the resulting oscillatory motion is given by a function of the same type as that of the impressed force, but differs from it in that its amplitude is greater in the ratio  $\mu:1$  and its phase different by the angle  $\omega \delta$ . Equation (9) represents the steady asymptotic motion after the superimposed free vibrations, which are damped, have died down.

Having found a particular solution of (4) we may write down the general solution immediately, say, in the form

$$u(t) = A e^{-\frac{r}{2m}t} \cos \nu(t - \delta) + c \mu e^{i\omega(t - \delta)}.$$

Again, a prescribed initial state will determine values of the constants  $A$  and  $\delta$ .

#### 4. General External Force $f(t)$

We now consider the case of a more general external force  $f(t)$  than that given by the simple exponential form above. The only assumptions we make concerning the function  $f(t)$  are that it be piece-wise continuous and smooth and that it be periodic, say, with period  $2\sigma$ . It is then possible to reduce the problem to the simple case of section 3.

Because of the assumptions made concerning  $f(t)$  it is possible to represent it by a Fourier series\*

$$(10) \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{\sigma}t + b_n \sin \frac{n\pi}{\sigma}t \right)$$

or, in the more concise complex form,

$$(10') \quad f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{\sigma}t}.$$

---

\* See Courant: Diff. and Int. Calculus, Vol. 1, Chap. IX, for a detailed discussion of Fourier series.



The coefficients  $a_n$ ,  $b_n$ ,  $c_n$ , are given by the following integrals

$$a_n = \frac{1}{\sigma} \int_0^{2\sigma} f(t) \cos \frac{n\pi}{\sigma} t \, dt, \quad b_n = \frac{1}{\sigma} \int_0^{2\sigma} f(t) \sin \frac{n\pi}{\sigma} t \, dt,$$

$$c_n = \frac{1}{2\sigma} \int_0^{2\sigma} f(t) e^{-\frac{in\pi}{\sigma} t} \, dt.$$

By the principle of superposition we need only find particular solutions for each of the non-homogeneous equations whose right-hand members are single terms of the series (10'). Thus we consider the set of equations

$$\mu u + r \dot{u} + ku = c_n e^{\frac{in\pi}{\sigma} t}, \quad (n = \pm 1, \pm 2, \pm 3, \dots).$$

The solutions are

$$u = c_n \mu_n e^{\frac{in\pi}{\sigma} (t - \delta_n)}.$$

The particular solution of (4) with  $f(t)$  given by (10') is then

$$(11) \quad u(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{\mu_n} e^{\frac{in\pi}{\sigma} (t - \delta_n)}.$$

Exercise: Calculate the distortion factors  $\mu_n$  and the phase displacements  $\delta_n$ .

In the event that  $f(t)$  is not a periodic function, we may obtain a particular solution of (4) by means of Fourier integrals,\* provided that  $f(t)$  tends to zero sufficiently fast as  $t \rightarrow \infty$ . We shall not carry out the details here but rather leave them for the following chapter where this method will again come up. We might mention that under rather general conditions  $f(t)$  may be represented in the form

$$f(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} g(\tau) e^{it\tau} \, d\tau$$

where

$$g(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\tau} \, d\tau$$

---

\* See Courant: Diff. and Int. Calculus, Vol. II, pages 318 ff.



and this leads to the solution

$$u(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \frac{g(\tau)}{-m\tau^2 + it\tau + k} e^{it\tau} d\tau.$$

In conclusion we remark that the solution of the initial value problem, say,  $u(0) = \dot{u}(0) = 0$ , for the non-homogeneous equation with a general function  $f(t)$  as right-hand member, may be written in an entirely different form, i.e.,

$$(12) \quad u(t) = \frac{1}{mv} \int_0^t \sin v(t-\tau) f(\tau) d\tau.$$

Equation (12) represents forced vibrations starting from complete rest.

Solution (12) is obtained by the so-called "impulse method" or "method of variation of parameters". Later we shall return to similar forms of solutions for other equations of forced motions. We will then take up the general principles of constructing a solution by considering the superposition of single impulses.

Exercise: Verify that (12) actually solves the initial value problem, namely, that (12) is a solution of  $mu + r\dot{u} + ku = f(t)$  and that  $u(0) = \dot{u}(0) = 0$ . In case  $f(t)$  is periodic, identify (12) with the solution obtained by the method of Fourier series.

## 5. Energy Balance

In the motion of a mechanical system represented by the differential equation

$$(1) \quad mu + r\dot{u} + ku = 0,$$

the kinetic energy  $T$  and the potential energy  $V$  are given by

$$T = \frac{1}{2} m\dot{u}^2 \quad \text{and} \quad V = \frac{1}{2} ku^2.$$

If we multiply both sides of equation (1) by  $\dot{u}$  and integrate the resulting equation between the limits 0 and  $t$  we obtain

$$\frac{m}{2} \dot{u}(t)^2 + \frac{k}{2} u(t)^2 + \int_0^t r\dot{u}(\tau)^2 d\tau = \frac{m}{2} \dot{u}(0)^2 + \frac{k}{2} u(0)^2 = \text{const.}$$



Thus the equation of energy balance is

$$(13) \quad T + V + \int_0^t r \dot{u}(\tau)^2 d\tau = \text{const.}$$

If there is no damping ( $r=0$ ), equation (13) simply expresses the law of the conservation of energy. In general, if  $r \neq 0$ , we may say that, after a time  $t$  has elapsed, the total energy of the system has decreased by an amount

$$\int_0^t r \dot{u}(\tau)^2 d\tau.$$

In other words, this much energy has been dissipated, e.g., converted into heat.

In case of self-excited vibrations where  $r < 0$ , the equation of energy balance states the fact that external energy is absorbed by the system. Actually, in nature and in technical applications, such phenomena do not occur in a trivial manner. Examples of such phenomena are the flutter of an airplane wing and the vibrations of a Hunting governor.

## 6. Resonance Phenomena

In the investigation of the phenomenon of resonance in connection with forced vibrations we consider the square of the distortion factor  $\mu$  expressed as a function of the frequency  $\omega$  of the impressed force:

$$(14) \quad \mu^2 = G(\omega) = \frac{1}{(k - m\omega^2)^2 + r^2 \omega^2}$$

where  $k$ ,  $m$ ,  $r$ , are to be considered as parameters characterizing a given vibrating system. We recall  $\omega_0 = \sqrt{k/m}$ . It is the natural frequency of the free undamped system, i.e., when  $r=0$ . The actual frequency of the free system, however, is not  $\omega_0$  but  $\nu$ . We assume  $r^2 - 4mk < 0$ , for otherwise the motion would be aperiodic.

We saw that the quantity  $\mu$  is the ratio of the amplitude of a forced vibration system to the amplitude of the impressed force. As  $\omega \rightarrow \infty$ ,  $G(\omega)$  approaches zero, so that an impressed





force with high frequency causes the amplitude of a vibrating system to decrease to zero and, hence, causes the motion to die down. On the other hand if  $\omega = 0$ , then  $G(0) = 1/k$ . Thus a steady force of unit magnitude would yield a motion with constant amplitude  $1/k$ .

We can find a value of  $\omega$  between zero and infinity for which  $G(\omega)$  has a maximum value. We calculate  $G'(\omega)$  and set it equal to zero. We get

$$2m^2 \omega^2 = 2mk - r^2 .$$

Denoting by  $\omega_1$  the value of  $\omega$  which satisfies this equation we see that  $\omega_1$  exists only if

$$2mk - r^2 \geq 0 ,$$

and is given by

$$(15) \quad \omega_1 = \sqrt{\frac{k}{m} - \frac{r^2}{2m^2}} = \sqrt{\omega_0^2 - \frac{r^2}{2m^2}} = \sqrt{v^2 + \frac{r^2}{4m^2}} .$$

We see that

$$\omega_1 < v < \omega_0$$

and

$$G(\omega_1) = \frac{1}{r^2 \left( \frac{k}{m} - \frac{r^2}{4m^2} \right)} = \frac{1}{r^2 v^2} .$$

Since  $G(\omega)$  is everywhere positive, increases monotonically in the neighborhood of  $\omega = 0$ , and tends to zero at infinity,  $G(\omega_1)$  is clearly a maximum of  $G(\omega)$ . The quantity  $\omega_1$  is called the resonance frequency of the system.

The graph of  $G(\omega)$  is called the resonance curve of the system. If the impressed frequency  $\omega$  has the value  $\omega_1$ , the amplitude of the system is a maximum and hence a state of resonance is attained. If  $r$  is small,  $\omega_1$  is close to the natural frequency  $\omega_0$ . Thus, in any system of forced vibrations, decreasing the damping factor  $r$ , while keeping  $m$  and  $k$  fixed, will make the phenomenon of resonance become more and more evident.



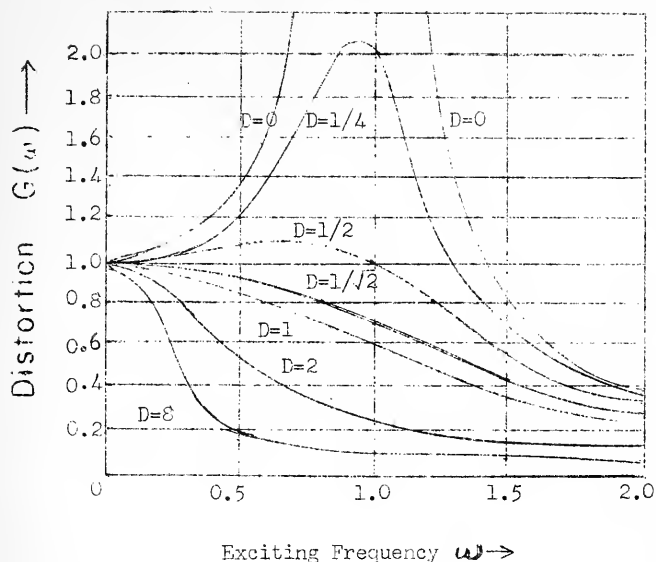


Fig. 8

In general, no resonance occurs if  $2mk - r^2 < 0$ . If  $2mk - r^2 = 0$ , the resonance curve is highest at  $\omega_1 = 0$  where  $G(0) = 1/k$ . Its tangent is horizontal there and remains almost horizontal for a while as  $\omega$  increases. Then, as  $\omega$  becomes large, the curve approaches zero asymptotically. In figure 8 there is sketched a family of resonance curves for  $m = k = 1$  (and, hence,  $\omega_0 = 1$ ), but for different values of  $D = r/2$ . If  $D$  is small, noticeable resonance occurs near  $\omega = 1$ . In the limiting case,  $D = 0$ ,  $G(\omega)$  has an infinite discontinuity at  $\omega = 1$ . As  $D$  increases, the resonance frequencies move to the left and, finally, for  $D = 1/\sqrt{2}$ ,  $\omega_1 = 0$ . For  $D > 1/\sqrt{2}$  no maximum exists and resonance is no longer present.

For undamped systems ( $r=0$ ) our solution fails when the exciting frequency  $\omega$  is equal to the natural frequency  $\omega_0$ , for then  $G(\omega)$  is infinite. Therefore we can not obtain a solution of

$$m\ddot{u} + ku = ce^{i\omega t}$$



in the form  $y e^{i\omega t}$ . However, a solution in the form  $y t e^{i\omega t}$  exists, where

$$y = \frac{c}{2im\omega}.$$

Thus, when resonance occurs in an undamped system we have the solution:

$$(16) \quad f(t) = \frac{c t e^{i\omega t}}{2im\omega} = \frac{c t e^{i\omega t}}{2i\sqrt{km}}.$$

Exercise: Find the solutions satisfying the initial conditions  $u(0) = \dot{u}(0) = 0$  for the following:

- a)  $u + 3\ddot{u} + 2u = \cos \omega t$
- b)  $u + \ddot{u} + u = \cos \omega t$
- c)  $u + \ddot{u} + u = \sin \omega t$
- d)  $2u + 2\ddot{u} + u = \cos \omega t$
- e)  $u + 4\ddot{u} + 4u = \cos \omega t$
- f)  $u + u = \cos t$

For the first four motions find also the amplitude, the phase, and the resonance frequency.

## 7. Recording Instruments

The purpose of such instruments as galvanometers, seismographs, oscillatory electrical circuits in radio receivers, and microphone diaphragms, is to reproduce or to record an oscillatory motion caused by some external periodic force. In such cases, the "displacement"  $u$  satisfies, at least in a first approximation, the differential equation

$$(4) \quad m\ddot{u} + r\dot{u} + ku = f(t)$$

where  $f(t)$  is the periodic force.

If the period of  $f(t)$  is  $2\sigma$  we can develop  $f(t)$  in a Fourier series\*

---

\* We will not concern ourselves here with questions of convergence; we merely assume all functions involved to be sufficiently regular as to conform to all the necessary convergence properties.



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi}{\sigma} t}.$$

By the principle of superposition the solution of the differential equation, apart from the free oscillations, is of the form

$$u(t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{\frac{i n \pi}{\sigma} t}$$

where

$$\gamma_n = c_n \mu_n e^{-\frac{i n \pi \delta_n}{\sigma}},$$

$$\mu_n^2 = \frac{1}{(k - m n^2 \frac{\pi^2}{\sigma^2})^2 + r^2 n^2 \frac{\pi^2}{\sigma^2}},$$

and

$$\tan \frac{\pi n}{\sigma} \delta_n = \frac{n r}{\sigma (k - m n^2 \frac{\pi^2}{\sigma^2})}.$$

We may describe the action of an arbitrary periodic external force  $f(t)$  as follows. We analyze the exciting force into simple periodic components each of which gives rise to a distortion of amplitude and phase displacement. The separate effects are then superimposed to give the total effect of the exciting force. If we are interested only in the amplitude distortion\*, a study of the resonance curve gives us complete information about the manner in which the motions of the recording apparatus reproduce the external exciting force. For very large values of  $n$  or of  $\omega (= \frac{n \pi}{\sigma})$ , the effect of the exciting frequency on the displacement  $u(t)$  will be very slight. On the other hand, exciting frequencies which are near to  $\omega_1$ , the resonance frequency, will effect  $u$  markedly.

In the construction of physical measuring and recording instruments the constants  $m$ ,  $k$ ,  $r$  are at our disposal, at least within wide limits. These should be chosen so that the shape

---

\* Phase displacement is only of secondary importance in applications since, e.g., it is imperceptible to the human ear. Moreover, it can be discussed in the same way as the amplitude distortion.





of the resonance curve conforms as well as possible to the special requirements of the measurements in question. Here two considerations predominate.

First of all, it is desirable that the apparatus be as sensitive as possible; that is, for all frequencies in question the quantity  $\mu$  should be as large as possible. We have seen that when  $\omega$  is small  $\mu$  is approximately proportional to  $1/k$ . Hence  $1/k$  is a measure of the sensitivity for small exciting frequencies. We may, therefore, increase the sensitivity by increasing  $1/k$ , i.e., by making the restoring force relatively weak.

The second important point is that the reproduced vibrations should be relatively free from distortion. Let us suppose that

$$\sum_{n=-N}^N c_n e^{\frac{i n \pi}{\sigma} t}$$

is a sufficiently good approximation of the exciting force  $f(t)$ . We then say that the instrument reproduces the exciting force with relative freedom from distortion if for all circular frequencies  $\omega \leq \frac{M\pi}{\sigma}$  the distortion factor  $\mu$  has approximately the same value. This condition is indispensable if we wish to derive accurate conclusions about the exciting force directly from the behavior of the instrument; if, e.g., a radio receiver is to reproduce both high and low musical notes with an approximately correct ratio of intensity.

The requirement that the reproduction be "distortionless" cannot be satisfied exactly, since no portion of the resonance curve is exactly horizontal. We can, however, attempt to choose the constants  $m$ ,  $k$ ,  $r$ , in such a way that the tangent to the resonance curve at  $\omega = 0$  is horizontal and remains approximately horizontal for small values of  $\omega$ . This insures that  $G(\omega)$  is approximately constant for small  $\omega$ . We saw in section 6 that we can get a horizontal tangent at  $\omega = 0$  by setting

$$(17) \quad 2mk - r^2 = 0 \quad .$$



Thus, given  $m$  and  $k$ , we may satisfy (17) by adjusting properly the friction  $r$  or, in an electrical circuit, the resistance  $\rho$ . The resonance curve then shows that for frequencies between 0 and the natural frequency  $\omega_0$  of the undamped system, the reproduced vibrations are relatively distortionless, and that for frequencies greater than  $\omega_0$  the damping is considerable. We therefore obtain relative freedom from distortion in a given interval of frequencies by first choosing  $m$  so small and  $k$  so large that the natural frequency  $\omega_0$  of the undamped system is greater than any of the exciting frequencies under consideration, and then choosing a damping factor  $r$  in accordance with equation (17).

## 8. Superposition of Different Vibrations with Random Phases

The simple problem of free vibrations with one degree of freedom leads to more profound questions than we might have expected. For example, suppose several sources produce vibrations of the same frequency  $\nu$  and amplitude  $h$ , but having different phases. Such would be the case with  $n$  different violins playing the same note. The resultant effect on a recording instrument would be a superposition of the individual effects. It is of interest to calculate the intensity of the superimposed vibrations and compare it with the intensity of the vibrations of a single violin. The intensity of a vibration is measured by the square of the amplitude, i.e.,  $h^2$ . If we calculate the resultant intensity it turns out to be  $nh^2$  whereas if the vibrations were all in phase it would quite naturally be  $n^2h^2$ . The question arises as to why this is so and to answer the question we proceed by considering the resultant effect of all the vibrations as a superposition of the single vibrations with phases distributed at random with equal probability.

A single vibration

$$h \cos \nu(t - \delta)$$



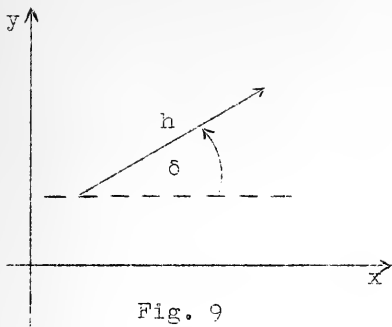


Fig. 9

may be represented geometrically by a vector of length  $h$  (the amplitude) making an angle  $\delta$  (the phase) with the positive x-axis of a coordinate system in a plane. (See figure 9). The superposition of two such vibrations would then be the vector sum of the vectors representing the two vibrations. If they are in phase the resultant amplitude would be  $2h$ , if not, then less than  $2h$ . In general,  $n$  vibrations superimposed would have a resultant effect represented by a vector joining the initial and terminal points (O and P in figure 10) of a polygonal path formed by the  $n$  vectors representing the  $n$  individual vibrations.

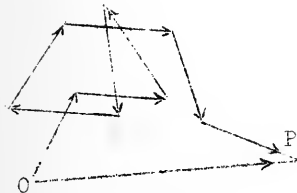


Fig. 10

We interpret the polygonal path from O to P as the "random walk" of an "intoxicated person" who, lacking all sense of direction, takes one step after another, each of length  $h$ . Looked at in this manner, the question then becomes: What is the probable distance  $OP = r$  from O of the drunken man after he has taken  $n$  steps? We shall content ourselves with obtaining the result by merely plausible means. We assume that the answer to the problem is the mean value of the quantity  $r^2$  although it must be noted that the mean value may not be the most probable value.

The rectangular coordinates of  $P(x,y)$ , if O is taken as the origin, are clearly

$$x = h \sum_{j=1}^n \cos \delta_j, \quad y = h \sum_{j=1}^n \sin \delta_j$$

where  $\delta_j$  is the direction angle made with the x-axis by the  $j$ -th step, or, in terms of violins, the phase of the  $j$ -th violin. Thus

The first part of the paper is devoted to a discussion of the general properties of the system. It is shown that the system is stable and that the solution is unique. The second part of the paper is devoted to a discussion of the asymptotic behavior of the system. It is shown that the system is asymptotically stable and that the solution converges to zero.



Figure 1. A graph showing the asymptotic behavior of the system. The curve starts at a point on the vertical axis and decreases towards the horizontal axis. The angle between the curve and the horizontal axis at the intersection point is labeled with the Greek letter theta (θ).

The third part of the paper is devoted to a discussion of the stability of the system. It is shown that the system is stable and that the solution is unique. The fourth part of the paper is devoted to a discussion of the asymptotic behavior of the system. It is shown that the system is asymptotically stable and that the solution converges to zero.

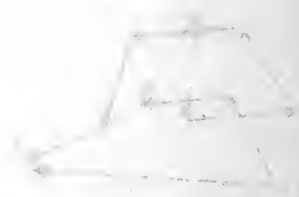


Figure 2. A graph showing the asymptotic behavior of the system. The curve starts at a point on the vertical axis and decreases towards the horizontal axis. The angle between the curve and the horizontal axis at the intersection point is labeled with the Greek letter theta (θ).

The fifth part of the paper is devoted to a discussion of the stability of the system. It is shown that the system is stable and that the solution is unique. The sixth part of the paper is devoted to a discussion of the asymptotic behavior of the system. It is shown that the system is asymptotically stable and that the solution converges to zero.

$$\overline{OP}^2 = r^2 = h^2 \left[ \left( \sum_{j=1}^n \cos \delta_j \right)^2 + \left( \sum_{j=1}^n \sin \delta_j \right)^2 \right]$$

whence the mean value of  $r^2$  is

$$\frac{h^2}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} \left[ \left( \sum_{j=1}^n \cos \delta_j \right)^2 + \left( \sum_{j=1}^n \sin \delta_j \right)^2 \right] d\delta_1 d\delta_2 \dots d\delta_n$$

The evaluation of the integral is simple. Squaring out the terms in the integrand yields  $n$  terms of the form

$$\cos^2 \delta_j + \sin^2 \delta_j = 1$$

and, in addition, product terms of the form

$$2\sin \delta_i \sin \delta_j \quad \text{and} \quad 2\cos \delta_i \cos \delta_j, \quad (i \neq j)$$

which when integrated from 0 to  $2\pi$  yield only zero. Hence the mean value of  $r^2$  is

$$\frac{h^2}{(2\pi)^n} \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} n d\delta_1 d\delta_2 \dots d\delta_n = nh^2$$

which is the value we stated to be the result.

Since the random walk problem is not only typical of a large class of statistical problems but also has many applications in other fields, a complete treatment of the problem will be given here for the more advanced student. At the same time this treatment will exemplify a general method of attacking not only differential equations but also the more general functional equations upon which additional conditions are imposed.

We first formulate and interpret the random walk problem in a somewhat more general way. Suppose a substance is distributed over the whole  $x,y$ -plane with a density  $w(x,y)$  and at distinct moments, say  $t = 1, 2, \dots, n, \dots$ , the whole substance is redistributed. (The discontinuity of the process is a simplifying idealization). We suppose the redistribution to be such that the substance at any one point is spread with equal





density along the circumference of a circle of radius  $h$  about this point. More precisely, if we denote the density at the point  $(x,y)$  at the time  $t = n$  by  $w_n(x,y)$ , then this function of three variables - two continuous space variables and a time variable  $n$  capable only of integral values - will satisfy the functional equation

$$(18) \quad w_{n+1}(x,y) = \frac{1}{2\pi} \int_0^{2\pi} w_n(\bar{x}, \bar{y}) d\theta$$

where  $\bar{x} = x + h \cos \theta$  and  $\bar{y} = y + h \sin \theta$  are the coordinates of a point on a circle of radius  $h$  about  $(x,y)$ . This is the functional equation for the random walk problem.

Exercise: Introduce in (18) the time variable  $t = n\tau$  where  $\tau$  is the actual time necessary for one redistribution. Show that if  $h \rightarrow 0$  and if it is assumed that  $\tau \rightarrow 0$  with the same order of magnitude as  $h^2$ , then (18) becomes in the limit the famous equation of heat flow:  $w_t - \Delta w = 0$ .

In order to determine the distribution  $w_n(x,y)$  in any specific case we must first know the initial distribution  $w_0(x,y)$ . Let this initial distribution be given by a prescribed function  $g(x,y)$ . First of all we notice that the solution  $w_n(x,y)$  is uniquely determined by (18) and the initial condition  $w_0(x,y) = g(x,y)$ . For if there were two different solutions, then the difference between them would also be a solution of (18) with the initial condition  $w_0 = 0$ . An examination of (18), however, shows that this implies  $w_1 = w_2 = \dots = w_n = 0$  so that the difference of the two solutions is zero and hence they coincide.

The random walk problem for the distribution of vibrations corresponds to the limiting case where the entire mass (suppose it to be unit mass) is concentrated at the origin initially. Thus, the initial density is infinite at the origin and zero elsewhere. Because of the singularity it is better to consider the unit mass as distributed initially in a small circle of radius  $\rho$  ( $\rho^2 = x^2 + y^2$ ) about the origin. Then  $w_0(x,y) = g(x,y)$  is equal to  $1/\pi\rho^2$  in this circle and zero outside the circle. After we have solved the problem with this initial condition, we may pass to the limit letting  $\rho \rightarrow 0$ .



The solution proceeds in a very typical manner which might be considered as a model for obtaining the solutions of similar problems.

1) We study the functional equation (18) in itself, looking for particular solutions. To this end we try to find solutions having the form

$$(19) \quad w_n(x, y) = f(n) \varphi(x, y) .$$

This is the so-called trial by "separation of variables" which we shall encounter frequently in the following chapters. This method yields an infinity of solutions of (18).

2) By superimposing some or all of these particular solutions we can construct a solution of (18) which also satisfies the initial condition  $w_0(x, y) = g(x, y)$ .

3) We then substitute our special initial function

$$g(x, y) = \begin{cases} 1/\pi \rho^2 & \text{for } x^2 + y^2 < \rho^2 \\ 0 & \text{for } x^2 + y^2 > \rho^2 \end{cases}$$

in this solution and let  $\rho \rightarrow 0$ , to obtain the solution to the problem of the random superposition of vibrations.

First we carry out step 1). Substituting (19) in (18) we find

$$\frac{f(n+1)}{f(n)} = \frac{\frac{1}{2\pi} \int_0^{2\pi} \varphi(\bar{x}, \bar{y}) d\theta}{\varphi(x, y)} .$$

Since the left side of this equation is a function of  $n$  alone and the right side solely a function of  $x$  and  $y$ , the two members can be equal for all values of  $n, x, y$ , only if they are equal to the same constant  $\lambda$ . Thus

$$\frac{f(n+1)}{f(n)} = \lambda$$

whence

$$(20) \quad f(n) = \lambda^n$$

if we set  $f(0) = 1$ . Also, setting the right member equal to  $\lambda$  we have a new functional equation for  $\varphi(x, y)$ :



$$(21) \quad \varphi(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\bar{x}, \bar{y}) d\theta \quad .$$

Now (21) is a linear functional equation and, from experience with the analogous type of differential equation, one would be led to attempt to find an exponential solution. Thus we try the complex exponential function  $\varphi = e^{i(\alpha x + \beta y)}$  where  $\alpha$  and  $\beta$  are real. (A complex function is taken in order that we may obtain solutions bounded at infinity). Substituting this in (21) we see that it will be a solution if

$$(22) \quad \lambda = \frac{1}{2\pi} \int_0^{2\pi} e^{ih(\alpha \cos \theta + \beta \sin \theta)} d\theta \quad .$$

In other words,  $\varphi = e^{i(\alpha x + \beta y)}$ , for arbitrary  $\alpha$  and  $\beta$ , will be a solution of (21) only if  $\lambda$  has the above value, the so-called "eigenvalue". We can write the integral representation of  $\lambda$  in a much simpler form if we recognize that the integral is invariant under rotations of the  $\alpha, \beta$ -plane. We may, therefore, write the integral in the form

$$(22') \quad \lambda = \frac{1}{2\pi} \int_0^{2\pi} e^{ih r \cos \theta} d\theta = J_0(hr) \quad ,$$

where  $J_0(hr)$  is the Bessel function of zero order\* and  $r = \sqrt{\alpha^2 + \beta^2}$ . We have now found a solution of (18), i.e.,

$$(23) \quad w_n(x, y) = f(n) \varphi(x, y) = e^{i(\alpha x + \beta y)} J_0^n(hr)$$

where  $\alpha$  and  $\beta$  are arbitrary.

The fact that  $\alpha$  and  $\beta$  are arbitrary enables us to carry out step 2). We superimpose all the solutions (23) obtained by letting  $\alpha$  and  $\beta$  take on all values. We get

$$(24) \quad w_n(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\alpha, \beta) J_0^n(hr) e^{i(\alpha x + \beta y)} d\alpha d\beta \quad .$$

---

\* For a discussion of Bessel functions, see Karman and Biot: Mathematical Methods in Engineering, Chap. II.



The function  $A(\alpha, \beta)$  is arbitrary with the proviso that it die down strong enough at infinity to allow the integral to converge.

To satisfy the initial condition  $w_0(x, y) = g(x, y)$  we see, by letting  $n = 0$  in (24), that

$$(25) \quad w_0(x, y) = g(x, y) = \iint_{-\infty}^{\infty} A(\alpha, \beta) e^{i(\alpha x + \beta y)} d\alpha d\beta.$$

We must now determine  $A(\alpha, \beta)$  so that (25) is satisfied. Again we have encountered a functional equation, this time an integral equation. However, this is solved immediately by the Fourier integral theorem if we assume  $g(x, y)$  to be sufficiently smooth and to die down rather rapidly at infinity. The solution is

$$(26) \quad A(\alpha, \beta) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} g(u, v) e^{-i(\alpha u + \beta v)} du dv.$$

If we place this value of  $A$  in (24) we obtain a four-fold integral which represents the solution of the random walk problem with initial distribution  $g(x, y)$ :

$$(27) \quad w_n(x, y) = \frac{1}{4\pi^2} \iiint_{-\infty}^{\infty} g(u, v) e^{i[\alpha(x-u) + \beta(y-v)]} J_0^n(hr) du dv d\alpha d\beta.$$

Finally we perform step 3) by letting  $g(x, y)$  be our special initial condition

$$g(x, y) = \begin{cases} 1/\pi\rho^2 & \text{for } x^2 + y^2 < \rho^2 \\ 0 & \text{for } x^2 + y^2 > \rho^2 \end{cases}.$$

Equation (26) then becomes

$$A(\alpha, \beta) = \frac{1}{4\pi^2\rho^2} \iint_{\sigma < \rho} e^{-i(\alpha u + \beta v)} du dv,$$

where  $\sigma = \sqrt{u^2 + v^2}$  and the integral is taken over the interior of the circle  $\sigma = \rho$ . By introducing polar coordinates  $u = \sigma \cos \xi$ ,  $v = \sigma \sin \xi$ , and recalling the rotational invariance exhibited by (22), we may transform the integral above into





$$A(\alpha, \beta) = -\frac{1}{4\pi} \frac{1}{\rho^2} \int_0^\rho \sigma \left( e^{-1} r \cdot \cos \xi \, d\xi \right) d\sigma$$

$$= \frac{1}{2\pi} \frac{1}{\rho^2} \int_0^\rho \sigma J_0(\sigma r) \, d\sigma \quad .$$

(The function  $J_0(\sigma r)$  is an even function of  $\sigma r$ ). Finally we integrate the last integral by parts and obtain

$$A(\alpha, \beta) = \frac{J_0(\rho r)}{4\pi \rho^2} - \frac{r}{4\pi \rho^2} \int_0^\rho \sigma^2 J_0'(\sigma r) \, d\sigma$$

where

$$J_0'(z) = \frac{dJ_0(z)}{dz} \quad .$$

By applying the mean value theorem for the integral calculus we find, for an intermediate value of  $\sigma$ , say  $\bar{\rho}$ , between 0 and  $\rho$

$$A(\alpha, \beta) = \frac{J_0(\rho r)}{4\pi \rho^2} - \frac{r\rho}{12\pi \rho^2} J_0'(\bar{\rho} r) \quad .$$

We now perform the passage to the limit,  $\rho \rightarrow 0$ , yielding

$$(28) \quad A(\alpha, \beta) = \frac{1}{4\pi \rho^2}$$

since  $J_0(0) = 1$  and  $J_0'(0) = 0$ .

Hence, for a mass concentrated initially at the origin, the solution is given by substituting (28) in (24):

$$w_n(x, y) = \frac{1}{4\pi \rho^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_0^n(hr) \, e^{i(\alpha x + \beta y)} \, d\alpha \, d\beta \quad ,$$

or, upon introducing polar coordinates for  $\alpha$  and  $\beta$ :

$$w_n(x, y) = \frac{1}{4\pi \rho^2} \int_0^\infty r J_0^n(hr) \left[ \int_0^{2\pi} e^{i\rho r \cdot \cos \theta} \, d\theta \right] dr$$

and



$$(29) \quad w_n(x,y) = \frac{1}{2\pi} \int_0^\infty r J_0^n(hr) J_0(\rho r) dr$$

where  $r^2 = \alpha^2 + \beta^2$ ,  $\rho^2 = x^2 + y^2$ . Equation (29) then yields the density of distribution of a unit mass, concentrated at the origin, after  $n$  redistributions.

This is the complete mathematical solution of the random walk problem as we have formulated it. For those who have a knowledge of Bessel functions, there is no difficulty in confirming the fact that the mathematical expectation for the intensity of  $n$  vibrations with same amplitudes and frequencies is  $n$  times the intensity of one of them provided only that the number  $n$  is sufficiently large. The details of this problem are left as an exercise. However, the following re-statement of the problem may be mentioned as a useful hint: Show that the probability for the total mass to be in a circular ring of radii  $h\sqrt{n} + \epsilon$  and  $h\sqrt{n} - \epsilon$ , for any  $\epsilon > 0$ , can be made as close to 1 as desired by taking  $n$  large enough. (Since  $w_n(x,y)$  represents the "probability density", the probability, or the amount of mass in a region  $G$  of the  $x,y$ -plane is

$$\iint_G w_n(x,y) dx dy \quad ) \quad .$$

## Part II. Systems of Several Degrees of Freedom

### 1. Two Degrees of Freedom

We consider dynamical systems moving about a state of stable equilibrium. Of such systems a pendulum is the simplest example. Its motion is completely characterized by one quantity (which we may call a coordinate) given as a function of time, e.g., the angle which the pendulum arm makes with the a vertical line. (Figure 1 of Part I). Systems whose motion and position are characterized by  $n$  independent such quantities are systems of  $n$  degrees of freedom.



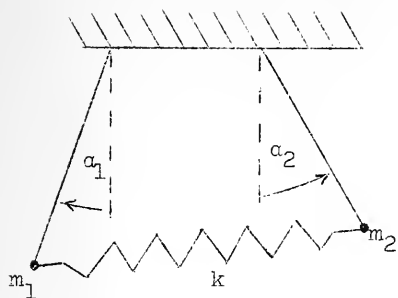


Fig. 1

The simplest example of such a system is the coupled double pendulum of figure 1. It consists of two single pendulums of the same length (say unit length) with respective masses  $m_1$  and  $m_2$ , coupled together by a spring with elastic constant  $k$ . The motion of the system is completely described by the two angles  $\alpha_1$  and  $\alpha_2$  given as functions of time. Thus this is a system of two degrees of freedom.

Another simple example of a system of two degrees of freedom is the electrical network with two meshes shown in figure 2.

A system of  $n$  degrees of freedom can be treated by considering its kinetic and potential energies together with a function representing the frictional forces present. In the case of the double pendulum (where we consider the oscillations so small that  $\sin \alpha$  may be replaced by  $\alpha$  and  $\cos \alpha$  by  $1 - \frac{1}{2} \alpha^2$ ), the kinetic energy  $T$  and the potential energy  $V$  are expressed by

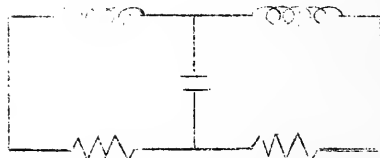


Fig. 2

$$(1) \quad \begin{cases} T = \frac{1}{2} m_1 \dot{\alpha}_1^2 + \frac{1}{2} m_2 \dot{\alpha}_2^2 \\ V = \frac{1}{2} m_1 g \alpha_1^2 + \frac{1}{2} m_2 g \alpha_2^2 + \frac{1}{2} k (\alpha_1 - \alpha_2)^2 \end{cases} .$$

We now briefly recall the usual treatment of a system of two degrees of freedom, in particular, that of the double pendulum. The problem is to determine the two coordinates  $\alpha_1$  and  $\alpha_2$  as functions of the time  $t$ . We consider only the case of



free vibrations (no damping). Later we will study the influence of frictional forces. The differential equations of motion (there are now two of them) for the double pendulum are

$$(2) \quad \begin{cases} m_1 \ddot{\alpha}_1 + m_1 g \alpha_1 + k(\alpha_1 - \alpha_2) = 0 \\ m_2 \ddot{\alpha}_2 + m_2 g \alpha_2 + k(\alpha_2 - \alpha_1) = 0 \end{cases}$$

where  $g$  is the acceleration due to gravity. Since the equations are linear and homogeneous we try to find solutions of the form

$$(3) \quad \alpha_1 = z_1 e^{i\omega t}, \quad \alpha_2 = z_2 e^{i\omega t}$$

where  $\omega$  is an unknown frequency and  $z_1$  and  $z_2$  are constants. Solutions of this form represent modes of harmonic vibrations. Substituting from (3) into (2) and factoring out the exponential function that occurs in both equations we have

$$(4) \quad \begin{cases} (m_1 g + k - m_1 \omega^2) z_1 - k z_2 = 0 \\ -k z_1 + (m_2 g + k - m_2 \omega^2) z_2 = 0 \end{cases}$$

These are two linear equations in the unknown quantities  $z_1$  and  $z_2$ . We wish them to have a solution other than the trivial one  $z_1 = z_2 = 0$ . To insure this we set the determinant of equations (4) equal to zero, i.e.,

$$\begin{vmatrix} m_1 g + k - m_1 \omega^2 & -k \\ -k & m_2 g + k - m_2 \omega^2 \end{vmatrix} = 0$$

Expanding the determinant yields a quadratic equation in the quantity  $\lambda = \omega^2$ :

$$(5) \quad m_1 m_2 \lambda^2 - (2m_1 m_2 g + m_1 k + m_2 k) \lambda + m_1 m_2 g^2 + (m_1 + m_2) g k = 0$$

which yields two roots  $\lambda_1 = \omega_1^2$  and  $\lambda_2 = \omega_2^2$ . After finding  $\omega_1$  and  $\omega_2$  we can solve the linear equations (4) by determining





the ratio  $z_1/z_2$  from either of them, getting two different results according as we substitute  $\omega_1$  or  $\omega_2$  in (4). Thus we obtain what are called the "natural frequencies"  $\omega_1$  and  $\omega_2$  and the modes of the natural vibrations expressed by the ratios  $z_1/z_2$  of the amplitudes. The most general vibration of the system is obtained as a linear combination or superposition of the two natural vibrations, and thus, by adjusting the two constants in the linear combination, we can satisfy given initial conditions, that is, we can find the specific motion for which the position and velocity of the two coordinates at time  $t = 0$  are arbitrarily prescribed.

In particular, if the two natural frequencies have almost the same value and if the two natural vibrations co-exist with approximately equal amplitudes, we obtain, by superposition, the phenomenon of beats. A typical example of this is given by our double pendulum when the two masses  $m_1$  and  $m_2$  are equal and the spring force  $k$  relatively weak. For instance, consider the vibration for which at time  $t = 0$  the first pendulum is at rest, i.e.,  $\dot{\alpha}_1 = 0$ , while the second has the initial position  $\alpha_2 = 0$  and the initial velocity  $\dot{\alpha}_2 = 1$ . The solution is given by

$$\begin{cases} \alpha_1 = \frac{1}{\omega_1} \cos\left(\frac{\omega_2 + \omega_1}{2}t\right) \sin\left(\frac{\omega_2 - \omega_1}{2}t\right) + \frac{1}{2}\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right)\sin\omega_2 t \\ \alpha_2 = \frac{1}{\omega_1} \sin\left(\frac{\omega_2 + \omega_1}{2}t\right) \cos\left(\frac{\omega_2 - \omega_1}{2}t\right) + \frac{1}{2}\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right)\sin\omega_2 t \end{cases}$$

where  $\omega_1 = \sqrt{g}$  and  $\omega_2 = \sqrt{g + 2k/m}$ . This represents a well known phenomenon which can be described briefly as follows. Each pendulum executes a fast vibration of frequency  $\frac{1}{2}(\omega_1 + \omega_2)$  with an amplitude which changes slowly with frequency  $\frac{1}{2}(\omega_2 - \omega_1)$ , (slowly because  $k$  is small). The two pendulums move in opposite phases, so that the amplitude of the vibrations of one reaches its maximum when the other is at rest. (See figure 3). Thus we have an oscillating transport of energy and motion between the two pendulums.



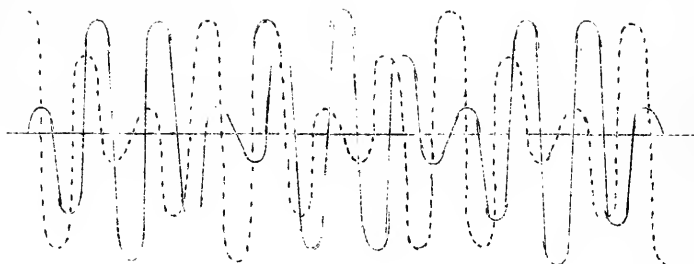


Fig. 3

## 2. Systems with $n$ Degrees of Freedom

To understand the general mathematical structure of problems with any number of degrees of freedom and to have a guiding principle for lines of attack in specific cases, one must think in more abstract terms. We consider any system

consisting of springs, mass points, connecting rods, etc., (or an electrical network consisting of any number of meshes), and characterized in its position and motion by  $n$  independent variable quantities (coordinates), which we denote by  $q_1, q_2, \dots, q_n$ . What they represent physically depends on the specific case. In electrical systems they may be the currents in the different meshes, in a mechanical system the coordinates of mass points, angles, or any combination of such simple quantities. For example, we may think of a multiple pendulum consisting of a chain of  $n$  simple pendulums with masses  $m_1, m_2, \dots, m_n$ ,

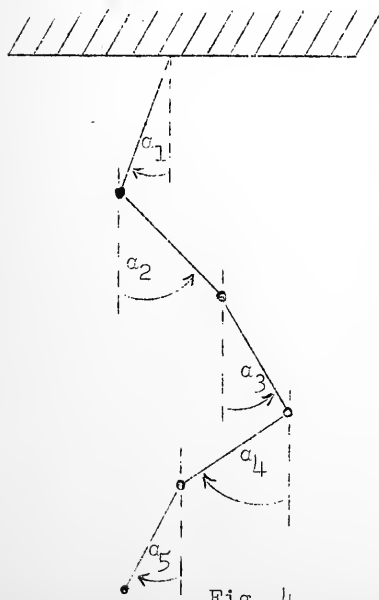


Fig. 4



and all having the same length  $\ell$ , as in figure 4. The simplest choice of coordinates here are the angles  $\alpha_1, \alpha_2, \dots, \alpha_n$ . Another example is the system composed of several distinct

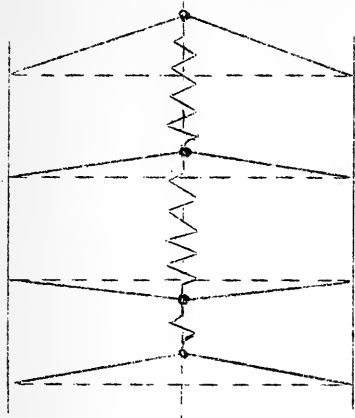


Fig. 5

elastic strings, parallel to each other in a plane and fastened at their endpoints. The masses of the strings are considered negligible, while each string carries at its center a concentrated mass  $m_1, m_2, \dots, m_n$ . The mass points are connected in series by springs of restoring force  $k$ . Then the quantities  $q_1, q_2, \dots, q_n$ , may denote the vertical deviations of the masses  $m_1, m_2, \dots, m_n$ , respectively, from their equilibrium positions.

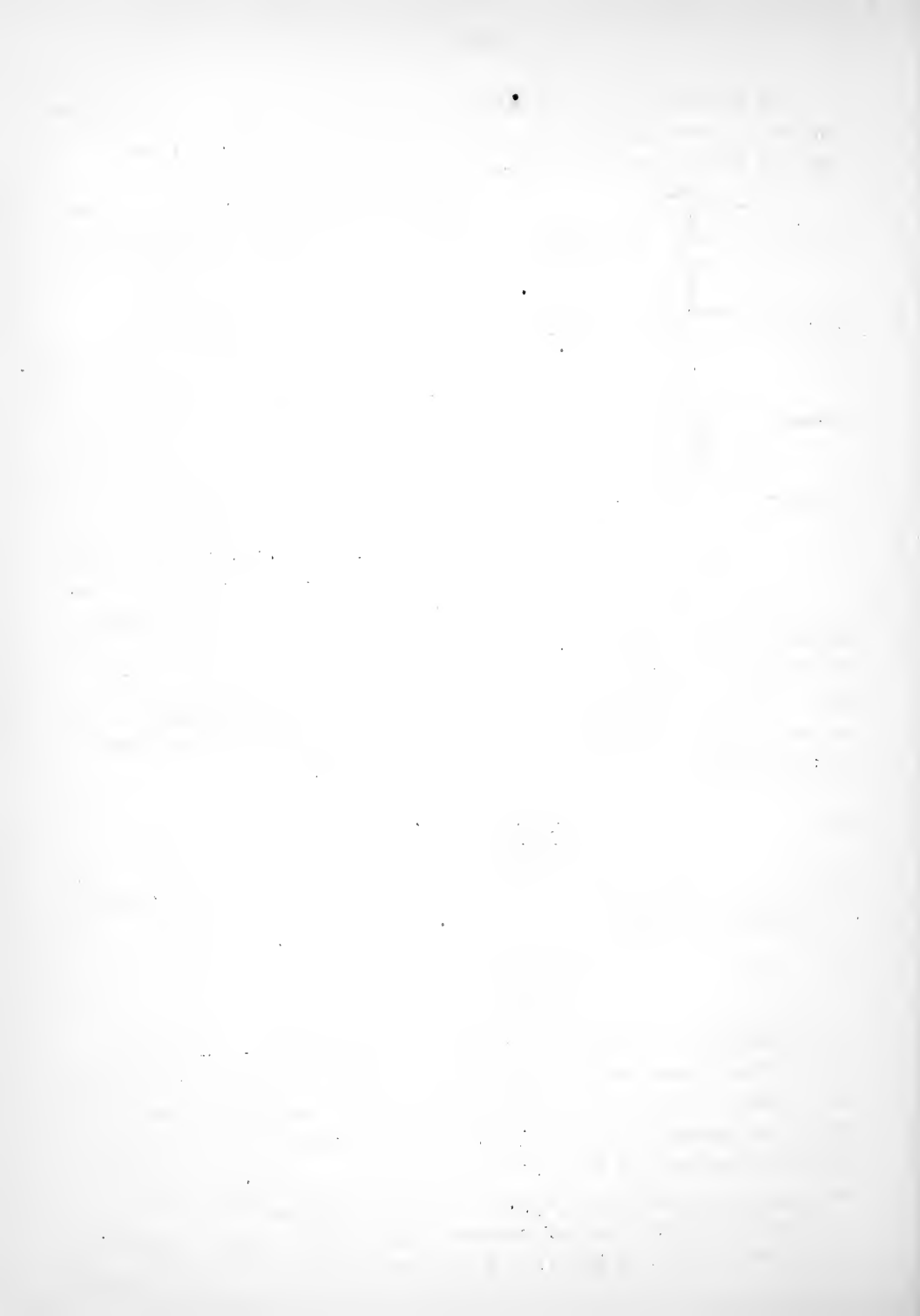
In the case of any such systems of  $n$  degrees of freedom (where, for the present, we neglect the influence of friction) the complete characterization of the motion is given by two functions, the kinetic energy  $T$  and the potential energy  $V$ , both of which are homogeneous quadratic expressions with constant coefficients. The kinetic energy has the form

$$(7) \quad T = \sum_{r,s=1}^n a_{rs} \dot{q}_r \dot{q}_s$$

which is a homogeneous quadratic form in the velocities  $\dot{q}$ , while the potential energy

$$(8) \quad V = \sum_{r,s=1}^n b_{rs} q_r q_s$$

is a quadratic form in the coordinates themselves. A stable equilibrium corresponds to fixed values  $q_r = 0$  of the coordinates, and we assume that our quadratic forms are positive except when all of the quantities  $q$  or  $\dot{q}$  vanish (in mathematical terms, the forms are assumed to be positive definite). Also, we may assume that  $a_{rs} = a_{sr}$  and  $b_{rs} = b_{sr}$ , since  $q_r q_s = q_s q_r$ , and we may, therefore, without loss of generality, put  $a_{rs} = a_{sr} =$  one-half the coefficient of  $\dot{q}_r \dot{q}_s$ , if  $r \neq s$ , etc.



We here have to rely on some preliminary knowledge of the subject, but it may be noted that the proof, e.g., of the form of  $T$  follows easily when we assume that the rectangular coordinates of the mass points of the system are linear functions of the  $q_r$ , and hence the velocities of the rectangular coordinates are linear expressions in the  $\dot{q}_r$ . Then we have only to express the kinetic energy as one-half the sum of masses times squares of velocities and, since each velocity is a homogeneous linear expression in the  $\dot{q}_r$ ,  $T$  will be a homogeneous quadratic expression in the latter quantities. As for potential energy we assume that it is a minimum for  $q_1 = q_2 = \dots = q_n = 0$ , as is stipulated by the supposed stable equilibrium. Expanding the potential energy in a Taylor series, the constant term, being irrelevant, may be set equal to zero, and the linear terms vanish because of the minimum statement. We assume that higher powers of the coordinates may be neglected in comparison with lower powers, so that, in the theory of small vibrations, the quadratic terms will be dominant and higher terms may be disregarded. This leads to an expression of the form (3) for the potential energy. For example, in the case of the multiple pendulum of figure 4,

$$T = \frac{\ell}{2} [m_1 \dot{\alpha}_1^2 + m_2 (\dot{\alpha}_1 + \dot{\alpha}_2)^2 + \dots + m_n (\dot{\alpha}_1 + \dots + \dot{\alpha}_n)^2]$$

$$V = \frac{S\ell}{2} [m_1 \alpha_1^2 + m_2 (\alpha_1 + \alpha_2)^2 + \dots + m_n (\alpha_1 + \dots + \alpha_n)^2]$$

while for the system of strings and springs considered above,

$$T = \frac{1}{2} \sum_{r=1}^n m_r \dot{q}_r^2$$

and

$$V = \frac{1}{2} \frac{S}{\ell} \sum_{r=1}^n q_r^2 + \frac{1}{2} k [(q_1 - q_2)^2 + \dots + (q_{n-1} - q_n)^2]$$

where  $S$  denotes the tension in each string and  $\ell$  the length of each string.

Exercise: Consider a massless elastic string tied between two fixed points and  $n$  loads of equal masses  $m$  distributed at equal distances along the string. What are the kinetic and potential









$$(13) \quad \begin{vmatrix} b_{11} - \lambda a_{11} & b_{12} - \lambda a_{12} & \dots & b_{1n} - \lambda a_{1n} \\ b_{21} - \lambda a_{21} & b_{22} - \lambda a_{22} & \dots & b_{2n} - \lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} - \lambda a_{n1} & b_{n2} - \lambda a_{n2} & \dots & b_{nn} - \lambda a_{nn} \end{vmatrix} = 0$$

where  $\lambda = \omega^2$ . This constitutes an equation of the  $n^{\text{th}}$  degree for the unknown quantity  $\lambda = \omega^2$ . In general the system yields  $n$  different solutions  $\omega_1, \omega_2, \dots, \omega_n$ , the natural frequencies of the system, and correspondingly  $n$  different values  $\lambda_r$ . We then substitute the  $\omega_r^2$  in (12) and solve for the ratios  $z_1/z_2, z_2/z_3$ , etc., characterizing the natural vibrations. These ratios are the natural (or, normal) modes of the natural vibrations.

It is clear that from a practical viewpoint this procedure will sometimes be very tedious. Since the determination of natural frequencies is of major importance in very many practical cases, we shall come back later to the question of practical methods.

The question of the natural vibrations of a system can also be handled in a second, slightly different, way, namely, by separating the variables. Just what we consider as coordinates of the system is theoretically immaterial, as long as everything can be expressed in terms of them. Now it is shown in the mathematical theory of quadratic forms that by a linear transformation of the form

$$q_r = \sum_{s=1}^n e_{rs} u_s, \quad ,$$

where the coefficients  $e_{rs}$  are constants, we can always bring the kinetic and potential energies into the following forms:

$$(14) \quad T = \sum_{r=1}^n \dot{u}_r^2$$

$$(15) \quad V = \sum_{r=1}^n \lambda_r u_r^2 \quad .$$



The coordinates  $u_r$ , replacing the  $q_r$ , are called the normal coordinates of the system. The actual finding of the coefficients  $e_{rs}$  of this transformation and the determination of the values of  $\lambda_r = \omega_r^2$  is, mathematically, equivalent to the procedure described above. The proof for the possibility of transformation to normal coordinates must be omitted here; it can be found in any algebra or general dynamics, e.g., Bocher: Introduction to Higher Algebra.

As soon as the energy functions are expressed in the simple forms (14) and (15) without product terms, the whole problem is solved; for then each coordinate  $u_r$  will vibrate independently of the others, since the Lagrange equations (9) become

$$(16) \quad \ddot{u}_r + \lambda_r u_r = 0 \quad , \quad (r = 1, 2, \dots, n)$$

and the vibrations are given by

$$(17) \quad u_r = v_r e^{i\omega_r t}$$

with  $\lambda_r = \omega_r^2$ . Any motion of the system may be represented by a linear combination of these natural vibrations.

In the special case of the double pendulum with two equal masses coupled by a spring (figure 1), the reduction of our energy functions to the separated forms is very simple. In this case we have

$$(1) \quad \begin{aligned} T &= \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) \\ V &= \frac{1}{2} mg(x_1^2 + x_2^2) + \frac{1}{2} k(x_1 - x_2)^2 \end{aligned}$$

If we put  $w_1 = x_1 - x_2$  and  $w_2 = x_1 + x_2$ , we have  $x_1 = \frac{w_1 + w_2}{2}$ ,  $x_2 = \frac{w_2 - w_1}{2}$ , so that

$$(18) \quad \begin{aligned} T &= \frac{m}{4} (\dot{w}_1^2 + \dot{w}_2^2) \\ V &= \frac{mg}{4} (w_1^2 + w_2^2) + \frac{1}{2} k w_1^2 \end{aligned} \quad ,$$



Letting  $u_1 = \frac{1}{2}\sqrt{m} w_1$ ,  $u_2 = \frac{1}{2}\sqrt{m} w_2$  ,

$$T = \dot{u}_1^2 + \dot{u}_2^2$$

(19)

$$V = (g + 2k/m)u_1^2 + gu_2^2 .$$

Thus the normal coordinates are

$$u_1 = \frac{1}{2}\sqrt{m}(\alpha_1 - \alpha_2)$$

$$u_2 = \frac{1}{2}\sqrt{m}(\alpha_1 + \alpha_2)$$

and the normal frequencies,

$$\omega_1 = \sqrt{g + 2k/m}$$

$$\omega_2 = \sqrt{g} .$$

### 3. Influence of External Forces

It is clear that after the separation of the variables, the influence of external forces can be investigated by the methods used for one degree of freedom, because the various natural vibrations behave exactly like independent vibrations. The phenomenon of resonance will occur whenever periodic external forces have frequencies near a natural frequency.

### 4. Influence of Friction

The influence of friction, however, is a very much more delicate matter and can only be touched upon very briefly here. The effect of frictional or damping forces is proportional to the velocities of the coordinates, and the most general assumption concerning it is that every coordinate interacts with every other coordinate. Assuming reciprocity of this interaction, the friction force may be expressed in terms of what Lord Rayleigh called a "dissipative function":

$$(20) \quad F = \sum_{r,s=1}^n c_{rs} \dot{q}_r \dot{q}_s , \quad (c_{rs} = c_{sr} = \text{const.})$$





by which equations (10) become

$$(21) \quad \sum_{s=1}^n (a_{rs} \ddot{q}_s + b_{rs} \dot{q}_s + c_{rs} q_s) = 0, \quad (r = 1, 2, \dots, n)$$

where the frictional term is just the partial derivative of  $\frac{1}{2}F$  with respect to the velocity  $\dot{q}_r$ . Introducing normal coordinates  $u_r$  corresponding to the system without friction, we obtain simplified equations of the form

$$(22) \quad \ddot{u}_r + \lambda_r u_r + \sum_{s=1}^n c_{rs} \dot{u}_s = 0 \quad (r = 1, 2, \dots, n)$$

where, of course, the coefficients  $c_{rs}$  have different values from those in (21). We see that a complete separation of the coordinates is no longer possible. A trial of the form

$$(23) \quad u_r = v_r e^{\mu t}$$

now leads to the system of equations

$$(24) \quad (\mu^2 + \lambda_r) v_r + \mu \sum_{s=1}^n c_{rs} v_s = 0$$

and the corresponding equation

$$\begin{vmatrix} \mu^2 c_{11} + \mu^2 + \lambda_1 & \mu c_{12} & \dots & \mu c_{1n} \\ \mu c_{21} & \mu^2 c_{22} + \mu^2 + \lambda_2 & \dots & \mu c_{2n} \\ \dots & \dots & \dots & \dots \\ \mu c_{n1} & \mu c_{n2} & \dots & \mu^2 c_{nn} + \mu^2 + \lambda_n \end{vmatrix} = 0.$$

The theory of the damped (or, sometimes, self-excited) vibrations of such a general system depends on the study of this equation. It is of the degree  $2n$  in  $\mu$ , and has complex roots which are conjugate in pairs. The theory of such systems, and the theory of stability and instability connected with it, must be omitted here, but will be briefly treated in a supplement to these notes.



An example of a very practical two degree of freedom system in which an external force and a damping force are present is the shock absorber shown in figure 6. It consists of a main mass  $M$  and an auxiliary mass  $m$ . An external force  $F$  is applied on  $M$  and a damping force (indicated by the dashpot) of strength  $r$  exists between  $M$  and  $m$ . It is desired that the amplitude of the motion of  $M$  should be as small as possible. This is accomplished by adjusting the mass  $m$ , the spring constants  $K$  and  $k$ , and the damping force  $r$ , in such a way that  $M$  will remain relatively stationary while  $m$  vibrates in some manner.

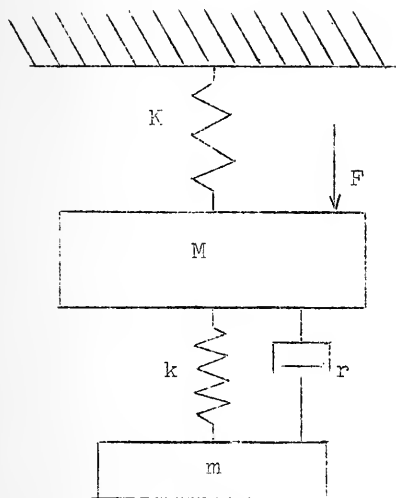


Fig. 6

Thus the mass  $m$  "absorbs" most of the "shock force" applied to  $M$ . For a complete discussion see Den Hartog: Mechanical Vibrations, pages 103 ff.

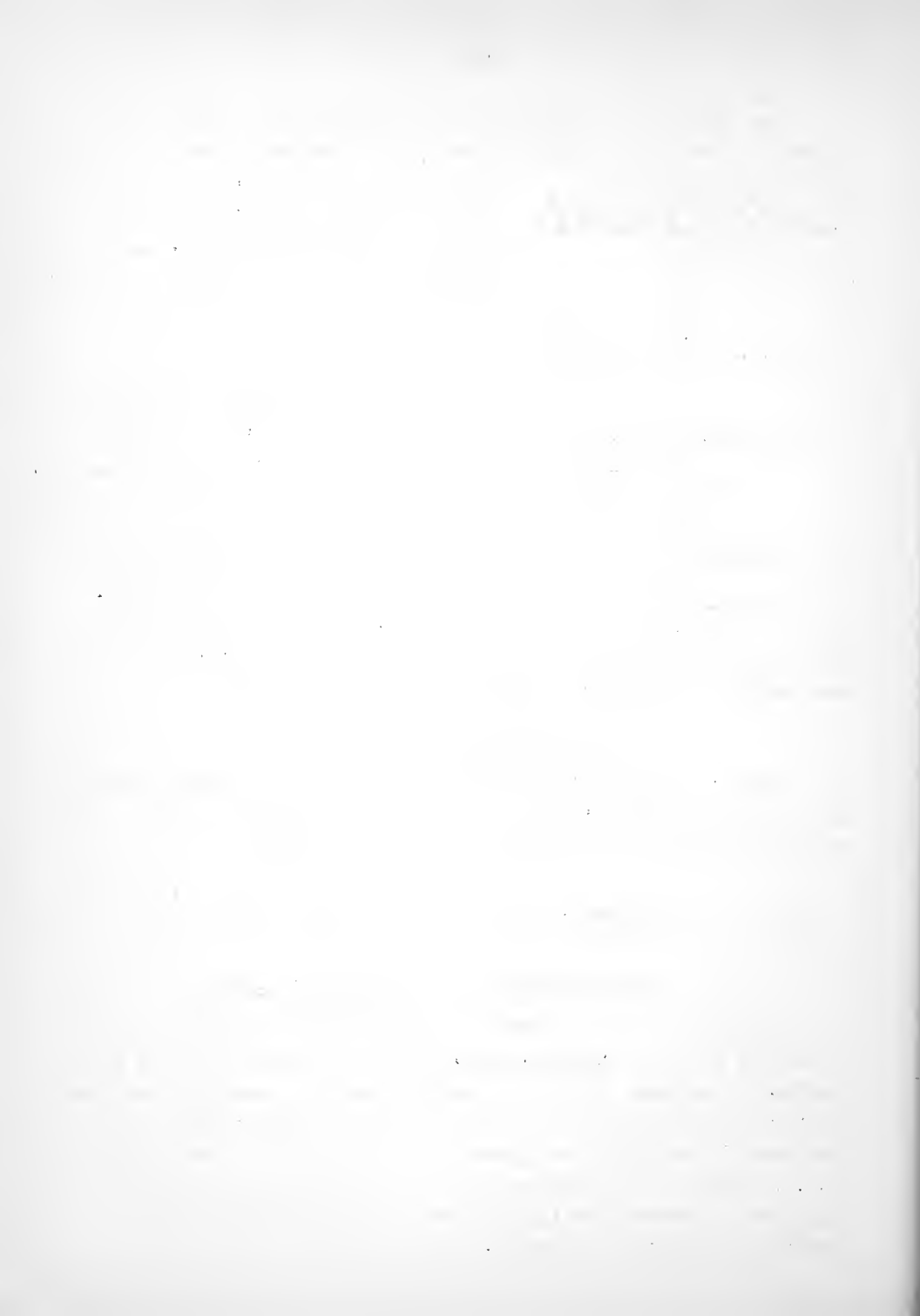
## 5. Energy Balance

An exceedingly important viewpoint for the understanding of resonance phenomena is that of the energy balance. An account of it will be given in a supplement to these notes.

### Part III. Extremum Properties of the Natural Frequencies and Modes

#### 1. The Frequencies as Successive Minima

We shall now show how the natural frequencies of a vibrating system  $\omega_1, \omega_2, \dots, \omega_n$ , can be characterized as the maximum or minimum values of certain variable quantities. Such a characterization will, as we shall see in Chapter Three, be exceedingly useful in the practical calculation of frequencies and, in addition, aid materially in the investigation of the behavior of natural frequencies when the vibrating system undergoes structural changes.



In order to make the mathematics as simple as possible, we will consider the system described by its normal coordinates  $u_1, u_2, \dots, u_n$ . The potential and kinetic energies, as we have seen, are then given by the equations

$$V = \sum_{r=1}^n \lambda_r u_r^2$$

$$T = \sum_{r=1}^n \dot{u}_r^2$$

Suppose now that the system is vibrating with a frequency  $\omega$  (not necessarily a natural frequency). Then

$$(1) \quad u_r = v_r e^{i\omega t}$$

where  $v_r$  is the amplitude of the corresponding  $u_r$ . The potential and kinetic energies above then become

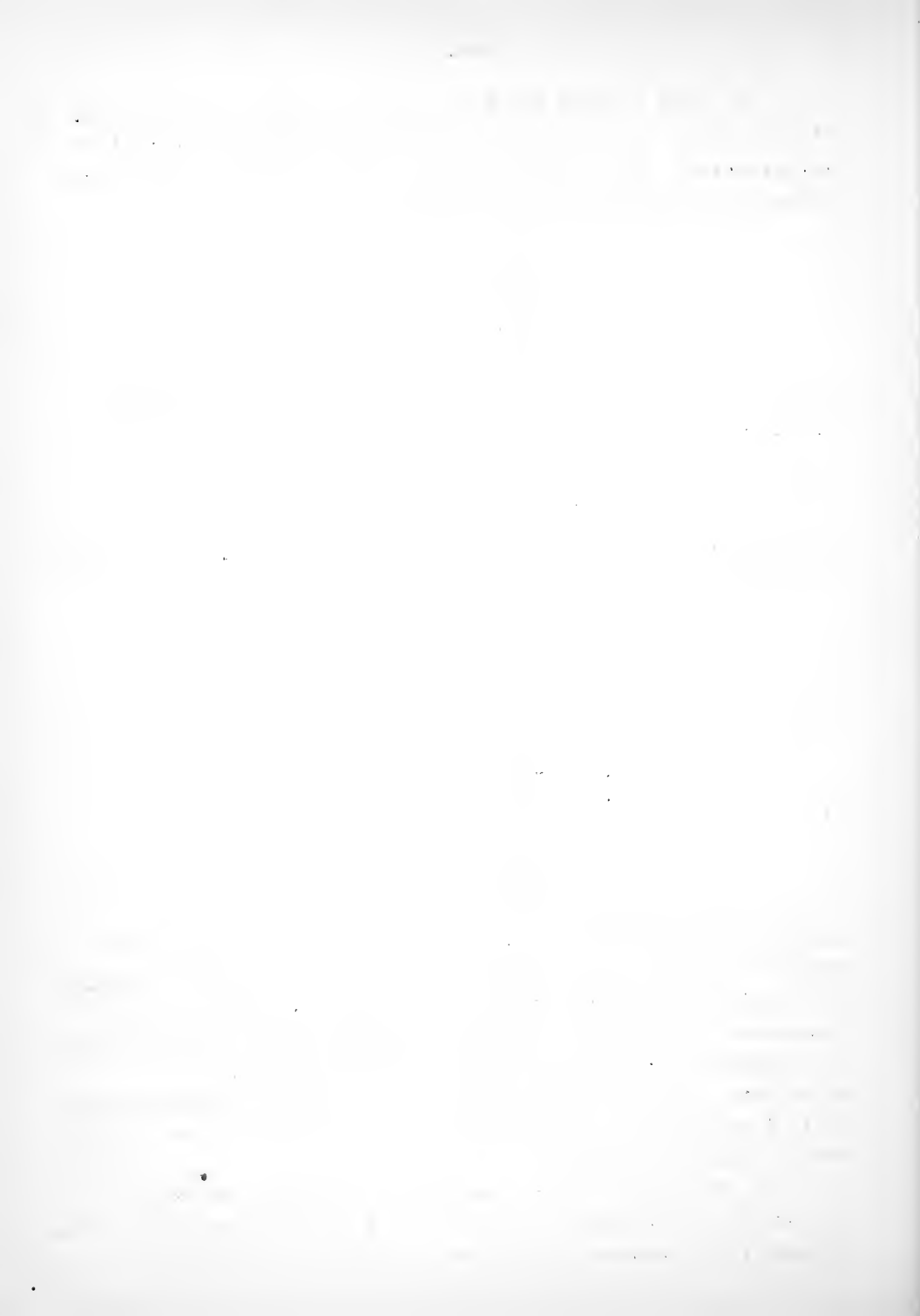
$$(2) \quad \begin{cases} V = \bar{V} e^{2i\omega t} \\ T = -\bar{T} \omega^2 e^{2i\omega t} \end{cases}$$

where

$$(3) \quad \begin{cases} \bar{V} = \sum_{r=1}^n \lambda_r v_r^2 \\ \bar{T} = \sum_{r=1}^n v_r^2 \end{cases}$$

(Incidentally,  $\bar{V}$  and  $\omega^2 \bar{T}$  are the mean values of  $V$  and  $T$  with respect to time  $t$ .) The quantities  $\bar{V}$  and  $\bar{T}$  we call the reduced potential and kinetic energies, respectively. They are purely mathematical quantities with no particular physical significance and represent the actual energy functions within factors depending on time and frequency. For the sake of simplicity we shall formulate the considerations that follow in terms of these reduced energies  $\bar{V}$  and  $\bar{T}$ .

In preparation for a discussion of certain extremum properties we arrange the quantities  $\lambda_r = \omega_r^2$  in increasing order of magnitude, assuming the subscripts to be so chosen that



$$(4) \quad \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \quad .$$

The corresponding frequencies  $\omega_r$  may then be called the fundamental tone, first overtone, second overtone, etc.

Because of the homogeneous character of the quadratic forms (3) we may introduce an unessential common factor into each of the  $v_r$  so that we may set

$$(5) \quad \bar{T} = \sum_{r=1}^n v_r^2 = 1 \quad .$$

It then follows immediately, on the basis of the inequalities (4), that

$$\bar{V} = \frac{n}{r=1} \lambda_r v_r^2 \geq \lambda_1 (v_1^2 + \dots + v_n^2) = \lambda_1 \bar{T}$$

or

$$(6) \quad \bar{V} \geq \lambda_1 \quad .$$

This inequality holds whatever values we choose for the amplitudes  $v_r$ . In particular, if we choose  $v_1 = 1, v_2 = v_3 = \dots = v_n = 0$ , then clearly

$$\bar{V} = \lambda_1 \quad .$$

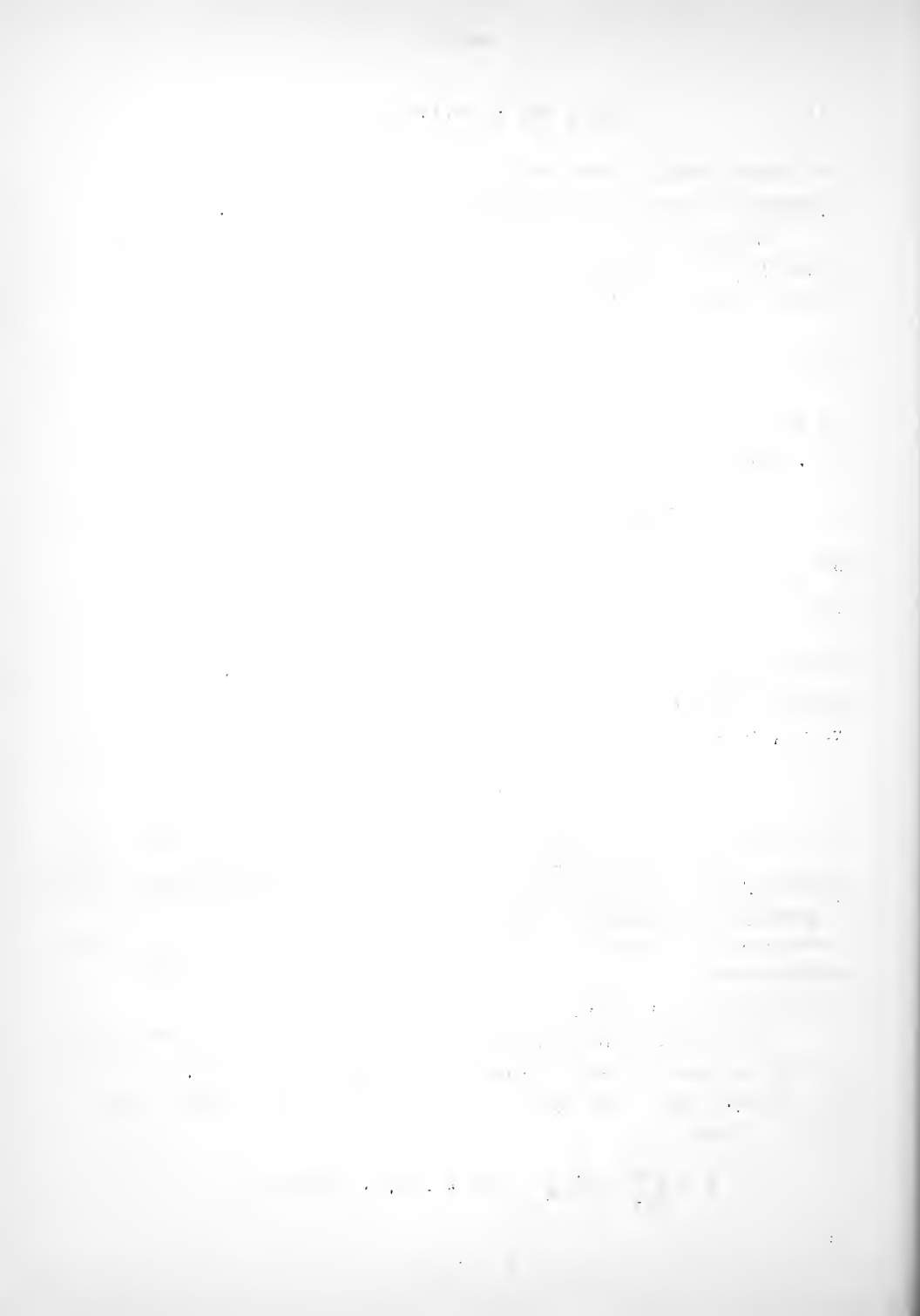
In other words, the first fundamental frequency  $\omega_1 = \sqrt{\lambda_1}$  is the square root of the minimum value of the reduced potential energy  $\bar{V}$  under the subsidiary condition that  $\bar{T} = 1$ . Because of the homogeneous character of  $\bar{V}$  and  $\bar{T}$  we could also say:  $\lambda_1$  is the minimum value of the quotient  $\bar{V}/\bar{T}$ , without any subsidiary conditions on the  $v$ 's.

By imposing other subsidiary conditions on the  $v$ 's we can, in like manner, characterize the remaining  $\lambda$ 's, i.e.,  $\lambda_2, \lambda_3, \dots, \lambda_n$ . For example, if  $\bar{T} = 1$  and if, in addition,  $v_1 = 0$ , then

$$\bar{V} = \sum_{r=1}^n \lambda_r v_r^2 \geq \lambda_2 (v_1^2 + \dots + v_n^2) = \lambda_2 \bar{T}$$

or

$$\bar{V} \geq \lambda_2$$





for all values of  $v_2, v_3, \dots, v_n$ , and, in particular,

$$\bar{V} = \lambda_2$$

for  $v_2 = 1, v_3 = v_4 = \dots = v_n = 0$ . Thus  $\lambda_2$  is the minimum value of  $\bar{V}$  under the subsidiary conditions  $\bar{T} = 1$  and  $v_1 = 0$ . In general,  $\lambda_k$  is the minimum value of  $\bar{V}$  under the conditions  $\bar{T} = 1$  and  $v_1 = v_2 = \dots = v_{k-1} = 0$ . In addition, this minimum is attained if we choose all the amplitudes equal to zero except  $v_k = 1$ .

## 2. Constraints - Frequencies as Maximum-Minimum

The characterization above, where, as mentioned, the condition  $\bar{T} = 1$  can be omitted if we consider the quotient  $\bar{V}/\bar{T}$  instead of  $\bar{V}$  alone, has the disadvantage that in order to characterize  $\lambda_k$  we have to have a preliminary knowledge of the preceding natural modes, i.e., it is a recursive process. For a deeper understanding of the behavior of the system an independent characterization of the frequencies, one which can be easily obtained as a consequence of the above, is more adequate. This characterization depends not upon minimum values of some quantity but upon certain maximum-minimum values. We consider first two examples to illustrate what we mean, the second of which is, as a matter of fact, closely connected with our problem.

First, consider two points A and B on different sides of a mountain range as in figure 1. Connecting A and B are an infinity of possible paths over the range, each of which has a point of maximum elevation. We then seek out that particular path for which the point of maximum elevation is a minimum. This minimum-maximum point, the point C

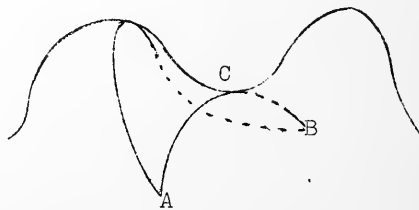


Fig. 1



in the figure, is called a "saddle-point". Incidentally, it is a point of unstable equilibrium under gravity.

The second example concerns an ellipsoid in three dimensions. The length of its semi-major axis is certainly the longest distance from the center of the ellipsoid to a point on it. Likewise, the semi-minor axis is the shortest such distance. Thus the first length is a maximum distance and the second a minimum. What, however, is the length of the medium principle axis? To characterize it we cut the ellipsoid with a plane through its center. The intersection formed is an "ellipsoid" of one dimension less, i.e., an ellipse. Consider the shortest axis of this ellipse and then move the plane of intersection in such a way that this shortest axis becomes as long as possible. The result will be the medium axis of the original ellipsoid. The latter is thus characterized as a maximum of a minimum or, shorter, a maximum-minimum. It is clear that it could also be obtained as a minimum-maximum.

Returning now to our dynamical system, we introduce the concept of a constraint. A constraint is any condition imposed upon the system that decreases the number of degrees of freedom by one. For example, the conditions  $v_1 = 0$  and  $v_1 = v_2$  are each constraints. In general, such a condition would be expressed by an equation of the form  $f(v_1, v_2, \dots, v_n) = 0$ . However, by expanding this expression in a Taylor series and supposing that higher powers of the coordinates may be neglected, we assume any constraint to be given by a linear homogeneous relation

$$(7) \quad h_1 v_1 + h_2 v_2 + \dots + h_n v_n = 0 \quad .$$

Physically, a constraint may be imposed on a dynamical system by fixing some particle originally free to move, or fixing an originally variable distance, etc. This can be achieved by letting masses increase to infinity or decrease to zero, or by replacing elastic connections by rigid ones (increasing spring forces to infinity). In any case, any specific constraint can be expressed by an equation of the form (7). The result of imposing a constraint is that the resulting system has only  $n-1$  degrees of freedom (and, hence, only  $n-1$  independent coordinates). Also, it

10. The first of these is the fact that the...

11. The second is the fact that the...

12. The third is the fact that the...

13. The fourth is the fact that the...

14. The fifth is the fact that the...

15. The sixth is the fact that the...

16. The seventh is the fact that the...

17. The eighth is the fact that the...

18. The ninth is the fact that the...

19. The tenth is the fact that the...

20. The eleventh is the fact that the...

21. The twelfth is the fact that the...

22. The thirteenth is the fact that the...

23. The fourteenth is the fact that the...

24. The fifteenth is the fact that the...

25. The sixteenth is the fact that the...

26. The seventeenth is the fact that the...

27. The eighteenth is the fact that the...

28. The nineteenth is the fact that the...

will have entirely different modes of vibration unless the constraint consists simply in fixing one of the normal coordinates, e.g., in the condition  $v_1 = 0$ .

In our previous examples we might consider constraints such as fixing the angle between a pair of rods in the multiple pendulum of figure 3, Part II, or replacing one of the strings of figure 4, Part II, by a rigid rod, etc.

Denote a given dynamical system by  $S$  and the system resulting from it by imposing a constraint by  $S'$ . If the squares of the natural frequencies of  $S'$  arranged in increasing order are denoted by  $\lambda'_1, \lambda'_2, \dots, \lambda'_n$ , then the following inequalities hold, i.e.,

$$(8) \quad \lambda_r \leq \lambda'_r \leq \lambda_{r+1} \quad .$$

In words: If a constraint is imposed upon a system  $S$  yielding a system  $S'$ , then the  $r^{\text{th}}$  natural frequency of  $S'$  will be greater than (or at least not less than) the  $r^{\text{th}}$  natural frequency of  $S$ . Also, it will be less than (or at least not greater than) the  $r+1$ -st natural frequency of  $S$ . Thus, in general, imposing a constraint on a system raises any given natural frequency to a value not greater than the next natural frequency of the original system.

A somewhat more general theorem holds if we impose not one but any number of constraints, say  $k-1$  of them:

$$(9) \quad \begin{cases} h_{11}v_1 + h_{12}v_2 + \dots + h_{1n}v_n = 0 \\ h_{21}v_1 + h_{22}v_2 + \dots + h_{2n}v_n = 0 \\ h_{k-1,1}v_1 + h_{k-1,2}v_2 + \dots + h_{k-1,n}v_n = 0 \end{cases}$$

where, of course,  $k-1 < n$ . The resulting system will then have  $n-k+1$  degrees of freedom. The theorem then states that the natural frequencies of  $S'$  satisfy the inequalities

$$(10) \quad \begin{cases} \lambda_r \leq \lambda'_r \leq \lambda_{r+k-1} & \text{for } r+k-1 \leq n \\ \lambda_r \leq \lambda'_r & \text{for } r+k-1 > n \end{cases} \quad .$$



These inequalities are a simple consequence of an important maximum-minimum property of the values  $\lambda_r$  expressed by the following fundamental theorem. We consider a system with  $k-1$  arbitrary constraints (9). Denote the minimum value of the quotient  $\bar{V}/\bar{T}$  by  $d(h)$  since it will depend upon the choice of the  $n(k-1)$  constants  $h_{rs}$ . Then we find a set of constants  $h_{rs}$  (i.e., a set of  $k-1$  constraints) such that  $d(h)$  becomes as large as possible. The maximum of  $d(h)$  thus obtained is the value  $\lambda_k$ . In other words, the  $k^{\text{th}}$  natural frequency of a vibrating system represents the highest pitch among the fundamental frequencies of all system of  $n-k+1$  degrees of freedom obtained from the given system by  $k-1$  constraints. This maximum-minimum is attained for the special constraints

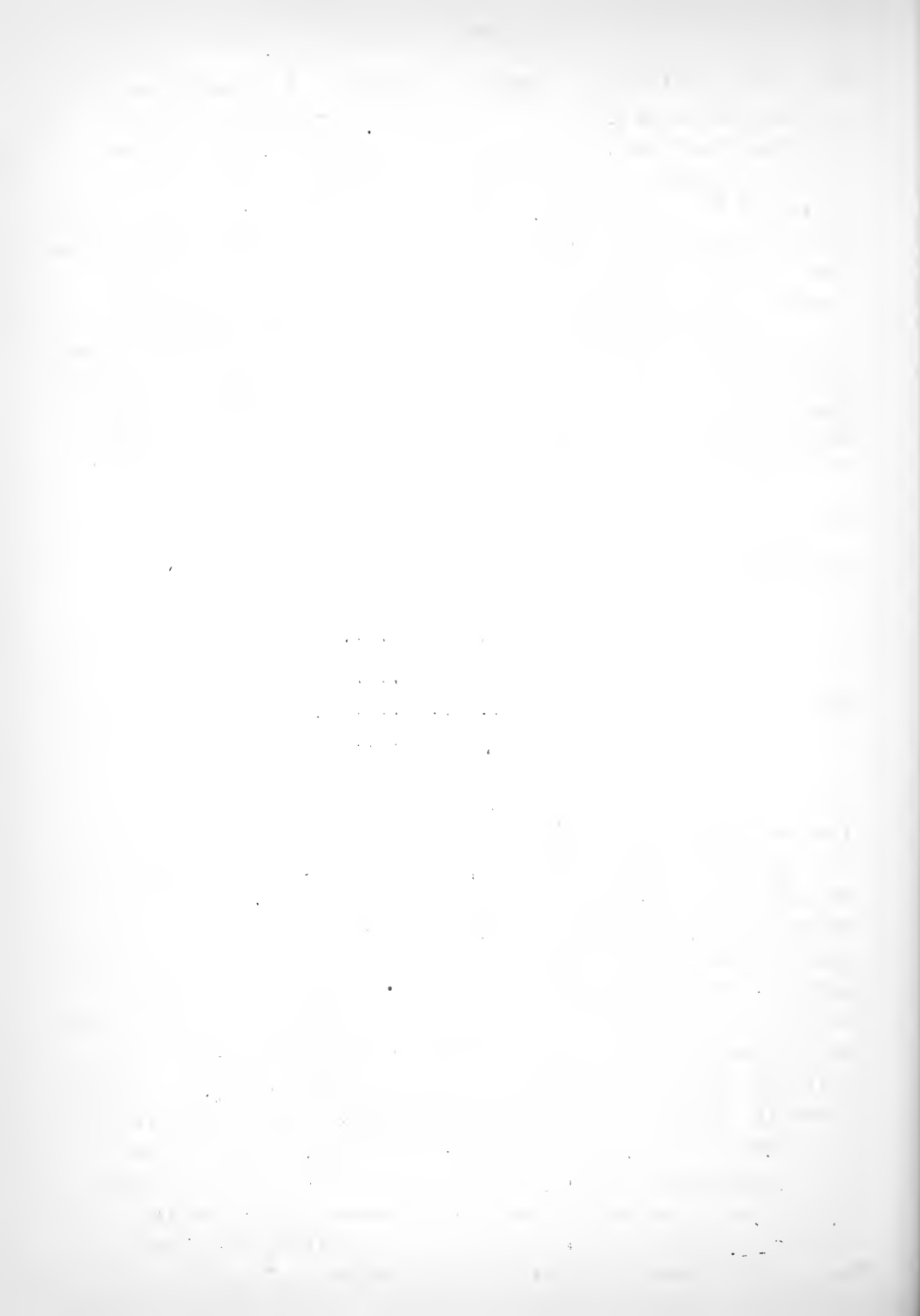
$$(11) \quad v_1 = v_2 = \dots = v_{k-1} = 0$$

for which the matrix of equations (9) is simply

$$(12) \quad \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

with  $k-1$  rows and  $n$  columns.

The mathematical proof of the theorem is very simple. From our recursive minimum property for  $\lambda_k$ , it is clear that  $d(h) = \lambda_k$  for the special system of constraints (11). Therefore we need only show that for any system of constraints of the form (9) the minimum of  $\bar{V}$  under the condition  $\bar{T} = 1$  does not exceed  $\lambda_k$ . To this end it suffices to find one single system of  $v_r$ 's satisfying (9) and making  $\bar{V} \leq \lambda_k$ , for then the minimum among all the admissible systems  $v_r$  will be all the more  $\leq \lambda_k$ . Such a special system is obtained by arbitrarily assuming  $v_{k+1} = v_{k+2} = \dots = v_n = 0$ . The system (9) of constraint conditions then becomes a system of  $k-1$  linear equations for  $k$  unknowns  $v_r$ . It is always possible to find values  $v_r$  satisfying such a system and then to multiply them all by a common factor so that they also satisfy the condition  $\bar{T} = 1$ . If we substitute this special system in  $\bar{V}$  we get





$$\bar{V} = \lambda_1 v_1^2 + \dots + \lambda_k v_k^2 \leq \lambda_k (v_1^2 + \dots + v_k^2)$$

or

$$(13) \quad \bar{V} \leq \lambda_k$$

which proves the theorem.

Exercise: Go through the above argument for the special cases  $k = 2$ ,  $k = 3$ .

The inequalities (10) are immediate consequences of this fundamental maximum-minimum theorem. To show this it suffices to consider the case of one constraint (7) and the inequalities (8). The quantity  $\lambda_r$ , belonging to the original system S, is the minimum value of  $\bar{V}/T$  under the  $r-1$  constraints

$$(14) \quad v_1 = v_2 = \dots = v_{r-1} = 0$$

according to the original minimum characterization of the  $\lambda_r$ , while  $\lambda'_r$  is the minimum under the constraints (14) and the additional constraint (7). Since this is clearly one of the admissible values under (14) alone, it cannot be less than the minimum value, whence

$$(15) \quad \lambda'_r \geq \lambda_r$$

On the other hand, according to our maximum-minimum theorem,  $\lambda_{r+1}$  is the highest value the square of the fundamental frequency can have under any  $r$  constraints, and therefore, in particular,  $\lambda'_r$ , the square of the fundamental frequency for the  $r-1$  constraints (14) and the additional constraint (7), cannot be greater than  $\lambda_{r+1}$ , i.e.,

$$(16) \quad \lambda'_r \leq \lambda_{r+1}$$

The inequalities (15) and (16) are just the relations given by (8), which is, therefore, proved. The generalization to (10) is clear.



### 3. Loss of Degrees of Freedom by Continuous Processes.

#### Behavior of the Frequency Spectrum Under Changes of the System.

The theorems of the last section lead immediately to the following consequences. Let  $S'$  be a system obtained from  $S$  by a stiffening process, that is, by increasing the spring forces of  $S$ , or, in general, by increasing the potential energy of  $S$  while keeping its kinetic energy fixed ( $\bar{V}' > \bar{V}$ ,  $\bar{T}' = \bar{T}$ ). It will then be true that all the frequencies  $\lambda_r'$  of  $S'$  are higher (or, at any rate, not lower) than the corresponding frequencies of  $S$ . Also, if the potential energy of  $S$  is decreased, then the frequencies will likewise decrease. Furthermore, if the potential energy of  $S$  is left unchanged but the kinetic energy altered, then the frequencies of  $S'$  will be smaller or greater than the corresponding frequencies of  $S$  according as the kinetic energy is increased or decreased. These theorems are of great importance in the investigation of vibrating systems in which changes of the spring force or masses are made.

The proofs in detail of the above statements are left as an exercise. The statements follow immediately from the maximum-minimum property because, e.g., if for any set of  $v$ 's the relation

$$(17) \quad \frac{\bar{V}}{\bar{T}} \leq \frac{\bar{V}'}{\bar{T}'}$$

holds, then the same inequality will subsist for minimum and maxi-minimum values of  $\bar{V}/\bar{T}$  and  $\bar{V}'/\bar{T}'$  under linear subsidiary conditions.

Energy changes of the type mentioned above can be used to produce the phenomenon of gradual "freezing" of degrees of freedom. We are interested in the behavior of the vibrating system in general when, by altering spring forces or masses, a degree of freedom of the system is "gradually" lost. Such a freezing may be effected by making the motion, or change, of a certain coordinate more and more difficult by attaching to such a change an increasingly large potential energy. For example, if we want to introduce a constraint of the form (7) not all at once but gradually, we may express this in terms of potential energy by constructing a new system in which the potential energy has the form



$$(18) \quad \bar{V}' = \bar{V} + s(h_1 v_1 + \dots + h_n v_n)^2 ,$$

where  $s$  is a large number ("spring force"). As we let  $s$  increase it becomes clear that the linear combination

$$h_1 v_1 + h_2 v_2 + \dots + h_n v_n$$

must decrease if we want our energy to remain with limited bounds. As  $s$  tends to infinity, the spring force tends to a rigid connection, and the rigid constraint (7) is the ultimate result.

As long as  $s$  remains finite the system retains  $n$  degrees of freedom but as  $s \rightarrow \infty$ , one degree "freezes". In the "spectrum" of natural frequencies this is expressed by the fact that  $\lambda_n' \rightarrow \infty$ , i.e., the highest vibration becomes infinitely fast as the spring force increases. This is in complete agreement with intuition.

Similarly, we may effect a freezing of a degree of freedom by increasing the kinetic energy which, for example, could be achieved by increasing the mass at a certain point. The effect is opposite to that above and the freezing will be reflected in the fact that the lowest natural frequency  $\omega_1' = \sqrt{\lambda_1'}$  will tend to zero.

It may be remarked, and left as an exercise for algebraically inclined students to investigate, that this behavior can be studied by algebraic manipulation of the determinant whose roots are  $\lambda_1, \lambda_2, \dots, \lambda_n$ . However, our method is more revealing of the true nature of the situation than such a complicated algebraic procedure.









frequencies of a slight change or "perturbation" in the structure of a vibrating system. Here  $S$  is the given system and  $S_\varepsilon$  is the perturbed system obtained from  $S$  by changing, in amounts proportional to  $\varepsilon$ , some of the quantities which define  $S$ .

Perhaps the best method of explaining the method of perturbations is to consider a simple but typical example. Let the system  $S$  be a double pendulum with equal masses  $m$ , with

pendulum arms of lengths  $\ell_1$  and  $\ell_2$ , and with no spring joining the two masses (or, what is the same, with the two masses joined by a spring of restoring force  $k = 0$ ). We then form the system  $S_\varepsilon$  from the system  $S$  by changing the restoring force in the spring from  $k = 0$  to  $k = \varepsilon\sqrt{2m}$ , all other quantities remaining unchanged.

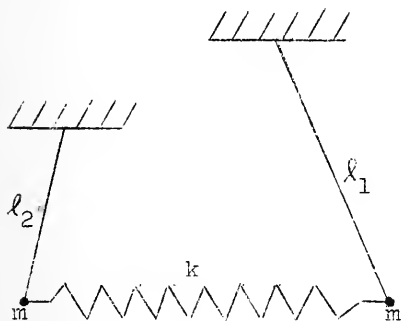


Fig. 1

The problem of finding the state of motion of  $S$  is simple.

We have

$$T = \frac{1}{2} m (\ell_1 \dot{\alpha}_1^2 + \ell_2 \dot{\alpha}_2^2) = \dot{u}_1^2 + \dot{u}_2^2$$

$$V = \frac{1}{m} mg (\ell_1 \alpha_1^2 + \ell_2 \alpha_2^2) = \frac{g}{\ell_1} u_1^2 + \frac{g}{\ell_2} u_2^2$$

where

$$u_1 = \ell_1 \sqrt{\frac{m}{2}} \alpha_1, \quad u_2 = \ell_2 \sqrt{\frac{m}{2}} \alpha_2.$$

Thus

$$\lambda_1 = \frac{g}{\ell_1} \quad \text{and} \quad \lambda_2 = \frac{g}{\ell_2},$$

whence  $\lambda_1 \neq \lambda_2$  if  $\ell_1 \neq \ell_2$ .

For  $S_\varepsilon$  the kinetic energy is the same as that for  $S$  but the potential energy differs by the addition of the term

$$\frac{k}{2} (\ell_1 \alpha_1 - \ell_2 \alpha_2)^2 = \frac{k}{\sqrt{2m}} (u_1 - u_2)^2 = \varepsilon (u_1 - u_2)^2.$$

Thus, for  $S_\varepsilon$ ,



$$T = \dot{u}_1^2 + \dot{u}_2^2$$

$$V = \lambda_1 u_1^2 + \lambda_2 u_2^2 + \varepsilon(u_1 - u_2)^2$$

or, in terms of reduced energies, (obtained by letting

$$u_r = v_r e^{i\omega t})$$

$$(2) \quad \begin{cases} \bar{T} = v_1^2 + v_2^2 \\ \bar{V} = (\lambda_1 + \varepsilon)v_1^2 + (\lambda_2 + \varepsilon)v_2^2 - 2\varepsilon v_1 v_2 \end{cases}.$$

(Notice that  $u_1$  and  $u_2$  are not the normal coordinates for the system  $S_\varepsilon$ ).

To obtain the natural vibrations  $\lambda'$  of  $S_\varepsilon$  we use equation (1a), namely,

$$\frac{\partial}{\partial v_j}(\bar{V} - \lambda'\bar{T}) = 0 \quad (j = 1, 2).$$

This yields the system of equations

$$(3) \quad \begin{cases} (\lambda_1 + \varepsilon - \lambda')v_1 - \varepsilon v_2 = 0 \\ -\varepsilon v_1 + (\lambda_2 + \varepsilon - \lambda')v_2 = 0 \end{cases}.$$

Let  $\lambda'_1$  and  $\lambda'_2$  be the two values of  $\lambda'$  (obtained by letting the determinant of (3) equal zero) for which (3) has a non-trivial solution. They are the squares of the natural frequencies of the perturbed system  $S_\varepsilon$ . Supposing  $\varepsilon$  to be small, we assume that

$$(4) \quad \begin{cases} \lambda'_1 = \lambda_1 + \varepsilon\mu_1 \\ \lambda'_2 = \lambda_2 + \varepsilon\mu_2 \end{cases}.$$

That is, we assume the squares of the natural frequencies of the perturbed system  $S_\varepsilon$  to differ by amounts proportional to  $\varepsilon$  from those of the system  $S$ . Equations (3) for  $\lambda' = \lambda'_1$  then become

$$(5a) \quad (1 - \mu_1)v_1 - v_2 = 0$$



$$(5b) \quad -\varepsilon v_1 + [\lambda_2 - \lambda_1 + (1 - \mu_1)\varepsilon]v_2 = 0 \quad .$$

Equation (5b) gives us the first natural mode of vibration

$$(6) \quad \frac{v_2}{v_1} = \frac{\varepsilon}{\lambda_2 - \lambda_1 + (1 - \mu_1)\varepsilon}$$

or, since  $\varepsilon$  is small and  $\lambda_2 \neq \lambda_1$ , it follows that  $(1 - \mu_1)\varepsilon \ll \lambda_2 - \lambda_1$ , whence we have, approximately.

$$(6a) \quad \frac{v_2}{v_1} = \frac{\varepsilon}{\lambda_2 - \lambda_1} \quad .$$

But from (5a) we get

$$\frac{v_2}{v_1} = 1 - \mu_1 \quad .$$

Hence

$$\mu_1 = 1 - \frac{\varepsilon}{\lambda_2 - \lambda_1}$$

from which it follows, by (4), that

$$(7) \quad \lambda_1' = \lambda_1 + \frac{\varepsilon^2}{\lambda_2 - \lambda_1} \quad .$$

In a similar manner the second mode of vibration is found to be

$$(8) \quad \frac{v_1}{v_2} = - \frac{\varepsilon}{\lambda_2 - \lambda_1}$$

and the second natural frequency

$$(9) \quad \lambda_2' = \lambda_2 + \varepsilon + \frac{\varepsilon^2}{\lambda_2 - \lambda_1} \quad .$$

Thus equations (6a) and (8) give us the natural modes and (7) and (9) the natural frequencies of  $S_\varepsilon$  all in terms of  $\varepsilon = k/\sqrt{2m}$  and the natural frequencies of  $S$ . These expressions hold, however, only if  $\varepsilon$  is small, i.e., if the spring force  $k$  is weak.



We see that, in accordance with the general principle that a stiffening produces an increase in frequencies, the frequencies of the perturbed system are higher than those of the original system.

It will be observed that in this treatment no evaluation of determinants was necessary; one linear equation gave us the corrected mode of vibration (i.e., the mode of the perturbed system  $S_\varepsilon$ ) and another the frequency correction. It is also worth pointing out that the corrections for  $\lambda$  automatically came out in terms of the first two powers of  $\varepsilon$ , while the mode of vibration only involves the first power. In general, if we carry this method out to any power of  $\varepsilon$ , we always obtain  $\lambda$  to a power higher by one than the power involved in the mode of vibration.

In the event that the two frequencies  $\lambda_1$  and  $\lambda_2$  of the unperturbed system are equal, we must make a slight modification in the argument, since we can no longer neglect terms in  $\varepsilon$  compared with  $\lambda_2 - \lambda_1$ . When  $\lambda_1 = \lambda_2$ , equations (5) become

$$(10a) \quad (1 - \mu)v_1 - v_2 = 0$$

$$(10b) \quad -v_1 + (1 - \mu)v_2 = 0 \quad .$$

For a non-trivial solution the determinant of this system must vanish, i.e.,

$$(11) \quad (1 - \mu)^2 - 1 = \mu(\mu - 2) = 0 \quad .$$

It has two roots  $\mu = 0$ ,  $\mu = 2$ . If we substitute  $\mu = 0$  in (10) we get, as the first normal vibration,

$$v_1 = v_2 \quad \text{and} \quad \lambda' = \lambda \quad .$$

Thus, the two pendulums swing parallel to each other so that the spring is not stretched and, therefore, exerts no influence on the frequency. Substituting  $\mu = 2$  in (10) we get the second normal vibration

$$v_1 = -v_2 \quad \text{and} \quad \lambda' = \lambda + 2\varepsilon \quad .$$





In this case the two pendulums alternately swing together, compressing the spring, and apart, stretching it. The frequency is thus increased.

It was necessary here to solve a determinant equation but, as it turned out, one that was easily factored. This example, even though it could be treated directly, is typical of the general method.

Exercise: Apply the perturbation method to a system of three strings of equal length and tension, each carrying a mass at its center, and each mass coupled to the other two by a weak spring. Consider the cases when all masses are different and when two are equal but different from the third.

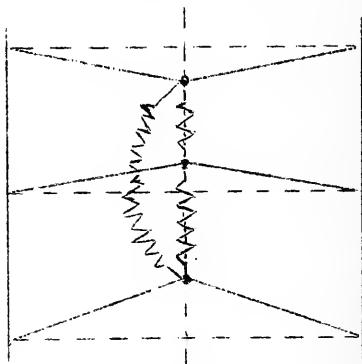


Fig. 2

## 2. The Method in General

It is relevant to write down the general formula giving the scheme of the perturbation method, although it is better in practice not to work deductively, but according to the pattern above. Suppose  $\bar{T}$  and  $\bar{V}$  for  $S$  are given in normal form

$$\bar{T} = \sum_{r=1}^n v_r^2$$

(12)

$$\bar{V} = \sum_{r=1}^n \lambda_r v_r^2 ,$$

while the perturbed system  $S_\epsilon$  is characterized by

$$\bar{T}_\epsilon = \bar{T} + \epsilon \sum_{r,s=1}^n \alpha_{rs} v_r v_s$$

(13)

$$\bar{V}_\epsilon = \bar{V} + \epsilon \sum_{r,s=1}^n \beta_{rs} v_r v_s .$$



In this new system  $S_\varepsilon$  the  $v$ 's are no longer normal coordinates and the  $\lambda$ 's no longer the squares of the natural frequencies. In the system  $S_\varepsilon$  the squares of the natural frequencies will be denoted by  $\lambda'_r$  and the corresponding modes of vibration will be given by systems  $v_1, v_2, \dots, v_n$  that differ slightly from a system where one coordinate alone is equal to 1 while the others are zero. The problem is to find the ratios  $v_1:v_2:v_3:\dots:v_n$  for the new natural modes and the corresponding  $\lambda'_r$  in terms of powers of  $\varepsilon$ , neglecting higher terms.

We assume that all the original frequencies of  $S$  are different:

$$(14) \quad \lambda_r \neq \lambda_s \quad \text{when} \quad r \neq s.$$

The equations of motion (1) for the system  $S_\varepsilon$  are

$$(15) \quad \begin{cases} [\lambda_1 - \lambda + \varepsilon(\beta_{11} - \lambda \alpha_{11})]v_1 + \varepsilon(\beta_{12} - \lambda \alpha_{12})v_2 + \dots + \varepsilon(\beta_{1n} - \lambda \alpha_{1n})v_n = 0 \\ \varepsilon(\beta_{21} - \lambda \alpha_{21})v_1 + [\lambda_2 - \lambda + \varepsilon(\beta_{22} - \lambda \alpha_{22})]v_2 + \dots + \varepsilon(\beta_{2n} - \lambda \alpha_{2n})v_n = 0 \\ \dots\dots\dots \\ \varepsilon(\beta_{n1} - \lambda \alpha_{n1})v_1 + \varepsilon(\beta_{n2} - \lambda \alpha_{n2})v_2 + \dots + [\lambda_n - \lambda + \varepsilon(\beta_{nn} - \lambda \alpha_{nn})]v_n = 0 \end{cases}$$

For the first natural mode of vibration,  $v_1$  will be nearly equal to 1 while all the other amplitudes will be small; let us assume  $v_2 = \varepsilon k_2, \dots, v_n = \varepsilon k_n$ . Furthermore, we assume that  $\lambda'_1$ , the square of the fundamental frequency of  $S_\varepsilon$ , is equal to  $\lambda_1 + \varepsilon \mu_1$ . Substituting  $\lambda = \lambda'_1$  in (15), these equations, with the exception of the first, become

$$(16) \quad \varepsilon(\beta_{r1} - \lambda_1 \alpha_{r1})v_1 + \varepsilon^2 \sum_{s=2}^n (\beta_{rs} - \lambda_1 \alpha_{rs})k_s + \varepsilon(\lambda_r - \lambda_1)k_r = 0$$

for  $r = 2, 3, \dots, n$ , since we may neglect  $\varepsilon \mu_1$  compared with  $\lambda_1$ . Neglecting the term in  $\varepsilon^2$  in (16), we get

$$(17) \quad \frac{v_r}{v_1} = \frac{\varepsilon k_r}{v_1} = -\varepsilon \frac{\beta_{r1} - \lambda_1 \alpha_{r1}}{\lambda_r - \lambda_1}.$$



The first equation of (15) we now use to calculate  $\lambda_1'$ . Substituting  $\lambda = \lambda_1 + \varepsilon \mu_1$  in it we get, neglecting higher powers of  $\varepsilon$ ,

$$(18) \quad \varepsilon(\beta_{11} - \lambda_1 \alpha_{11} - \mu_1) v_1 + \varepsilon^2 \sum_{s=2}^n (\beta_{1s} - \lambda_1 \alpha_{1s}) k_s = 0.$$

Again, if we neglect terms in  $\varepsilon^2$  as compared with terms in  $\varepsilon$ , we have

$$\mu_1 = \beta_{11} - \lambda_1 \alpha_{11}$$

whence

$$(19) \quad \lambda_1' = \lambda_1 + \varepsilon \mu_1 = \lambda_1 + \varepsilon(\beta_{11} - \lambda_1 \alpha_{11}).$$

More precisely, we find by using (17)

$$\mu_1 = \beta_{11} - \lambda_1 \alpha_{11} - \varepsilon \frac{\sum_{s=2}^n (\beta_{1s} - \lambda_1 \alpha_{1s})^2}{\lambda_s - \lambda_1}$$

which gives  $\lambda_1'$  up to terms in  $\varepsilon^2$ . In exactly the same way, find for the  $j^{\text{th}}$  mode of vibration of  $S_\varepsilon$ ,

$$(20) \quad \begin{cases} \frac{v_r}{v_j} = -\varepsilon \frac{\beta_{rj} - \lambda_j \alpha_{rj}}{\lambda_r - \lambda_j} & (r = 1, 2, \dots, j-1, j+1, \dots, n) \\ \lambda_j' = \lambda_j + (\beta_{jj} - \lambda_j \alpha_{jj}) \end{cases}.$$

If some of the original  $\lambda$ 's are equal, this procedure breaks down, but the method can be modified in a manner similar to that employed for the case of two degrees of freedom.



## CHAPTER TWO

### VIBRATIONS AND EQUILIBRIUM PROBLEMS OF SYSTEMS WITH INFINITELY MANY DEGREES OF FREEDOM

#### Introduction

The general concept of a system of a finite number of degrees of freedom as discussed in the previous chapter is rather abstract, inasmuch as no specifications concerning the visual structure of the system were made. On the other hand, by sacrificing the concreteness we were able to isolate and to exhibit the essential underlying features characterizing such general systems. A further idealization, however, often leads to a consideration of systems which more closely resemble actual physical systems than do the abstract systems of the last chapter. In many specific cases there exists a certain symmetry in the mechanical or electrical arrangement which suggests that a system of  $n$  degrees of freedom might advantageously be replaced by a system with infinitely many degrees of freedom, obtained by allowing  $n$  to increase without limit.

For example, consider a weightless elastic string, of unit length and with tension  $S$ , carrying  $N$  mass points placed at equal distances along it. Let each of these particles have mass  $m/N$  and suppose that they all move perpendicularly to the equilibrium position of the string and always in a fixed plane through this equilibrium position. Denoting the deviation of the  $r^{\text{th}}$  particle by  $q_r$ , the system is clearly one of  $N$  degrees of freedom with coordinates  $q_1, q_2, \dots, q_N$ .

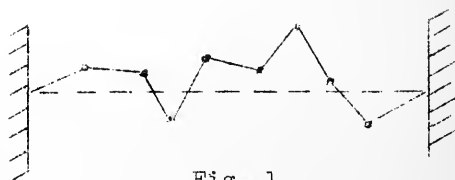


Fig. 1

As long as  $N$  is finite the problem of the state of vibratory motion of the system is one which can be treated by the methods of Chapter One. If, however, we let  $N$  increase without bound, the system of discrete mass points becomes, in the limit, simply





a string with uniform mass density  $m$  for which the number of degrees of freedom is infinite. Thus the problem of the vibrating continuous elastic string may be looked upon as the limiting case of a string composed of discrete mass particles, when the number of particles becomes infinitely large and the mass of each particle becomes proportionately small.

Consider, also, a network of  $N$  mass particles, each of mass  $m/N$ , arranged, as in figure 2, in such a manner that every particle is linked to each of its four neighbors by springs with the same restoring force. Now allow  $N$  to grow infinitely large and, simultaneously, let the area of each mesh approach zero. We get, in the limit, the case of a continuous elastic membrane with mass density  $m$ . In a similar manner, elastic cantilevers and plates may be considered as limiting cases of systems of a finite number of degrees of freedom, as this number increases without bound.

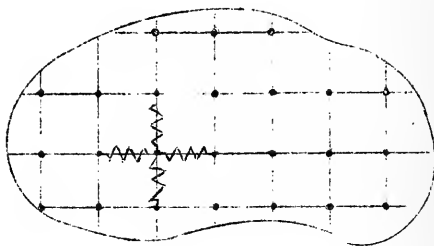


Fig. 2

Theoretically, such a limiting process is purely an idealization. Actually, a physical system may be considered as consisting of very many discrete atoms or other elements. However, it would be foolish to attempt to study the motions of each of the individual atomic particles simply to be in accord with the physical fact that the system has but a finite number (although a very large one) of degrees of freedom. As we shall see, the idealization obtained by passing to a limit offers a tremendous simplification and, furthermore, is really the proper basis for the actual treatment of such problems.

In passing we mention that there exist other physical systems in which the limiting process does not lead to anything like the continuous distributions arrived at above. A typical case is the electric filter chain which, in reality, consists of



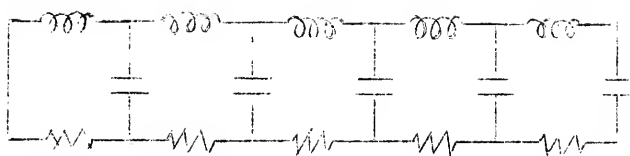


Fig. 3

but a finite number of distinct elements, as shown in figure 3. However, even in this case a great advantage is gained if we consider the chain as extending to infinity in either direction. We shall return to this system in a supplement and confine ourselves in this chapter to continuous systems.

## Part I. Equations of the String and Membrane

### 1. The String

In figure 4 is shown a stretched string fastened at the endpoints of the interval  $0 \leq x \leq \ell$  of the  $x$ -axis. We suppose the points of the string to lie always in the  $u, x$ -plane and to be subject to motion along lines perpendicular to the  $x$ -axis. The deviation of the string from its equilibrium position (the  $x$ -axis) is given by a function  $u = u(x)$  depending on  $x$ . The position of the string is known if we know the value of  $u$  for every  $x$  in the interval  $0 \leq x \leq \ell$ . This is in contrast to the case of systems of a finite

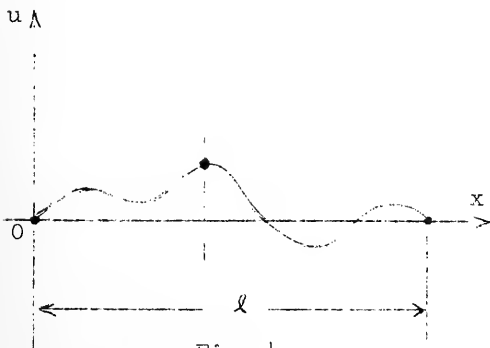


Fig. 4

number of degrees of freedom where a finite number of coordinates characterized the motion of the entire system. If we are interested, as we shall be, not only in the problem of the equilibrium of the string, but also in the problem of its motion,



then  $u$  is not only a function of  $x$  but also of the time  $t$ , i.e.,  $u = u(x, t)$ .

We denote by  $m$  the density of mass. In general it will not be uniform throughout the length of the string. If this is the case  $m$  is given as a function  $m = m(x)$  of the points  $x$  of the string and the total mass of the string is then  $\int_0^l m(x) dx$ . For our purposes at present, however, we shall assume that the string has uniform density, i.e.,  $m(x)$  is a constant. The total mass is then  $M = m l$ .

To obtain the differential equation characterizing the state of motion or equilibrium of the string we consider the position  $u(x, t)$  of the string at a fixed time  $t$ . Let

$x_1 \leq x \leq x_2$  be a small interval of length  $\Delta x = x_2 - x_1$  on

the  $x$ -axis. The assertion that the motion of each particle is in a line perpendicular to the  $x$ -axis may be expressed by the statement that the horizontal component  $S$  of the tension is a constant. Denoting by  $\alpha$

the angle between the positive  $x$ -axis and the tangent to the string at  $x = x_2$ , the vertical component of the tension at this point is seen to be  $S \tan \alpha$  or

$$S \left( \frac{\partial u}{\partial x} \right)_{x=x_2}.$$

In the same way the vertical component of the tension at  $x = x_1$  is found to be

$$-S \left( \frac{\partial u}{\partial x} \right)_{x=x_1}.$$

Hence the total elastic force on the segment of string in the interval  $\Delta x$  due to the vertical component of the tension is

$$S \left[ \left( \frac{\partial u}{\partial x} \right)_{x=x_2} - \left( \frac{\partial u}{\partial x} \right)_{x=x_1} \right]$$

for sufficiently small  $\Delta x$ .

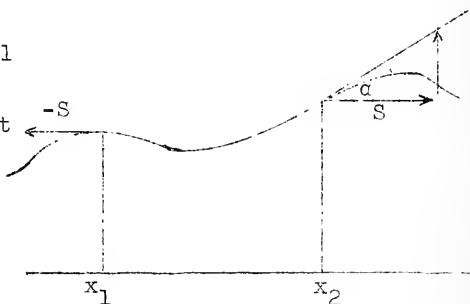


Fig. 5



On the other hand, this force must be in equilibrium with the inertial force and any external force that might be present. The inertial force is

$$\int_{x_1}^{x_2} m u_{tt} dx = m \bar{u}_{tt} \Delta x$$

where  $\bar{u}_{tt}$  is the value of  $u_{tt}$  for some point in the interval  $x_1 \leq x \leq x_2$ . Also, if the density of the external force is given by  $f(x,t)$ , then the force it exerts in our small interval is  $f(x,t) \Delta x$ . The equation expressing the equilibrium of the forces on the string in the interval is, thus

$$S \left[ \left( \frac{\partial u}{\partial x} \right)_{x=x_2} - \left( \frac{\partial u}{\partial x} \right)_{x=x_1} \right] = m \bar{u}_{tt} \Delta x + f(x,t) \Delta x .$$

Dividing by  $\Delta x$  and letting  $\Delta x \rightarrow 0$ , we get, in the limit,

$$(1) \quad S u_{xx} - m u_{tt} = f(x,t) , \quad (0 \leq x \leq \ell ; t \geq 0) ,$$

which is the general equation of motion of the elastic string.

If we are interested in the problem of equilibrium of the string we must assume that all functions which occur are independent of the time  $t$ , i.e., the impressed force  $f = f(x)$  and the deflection  $u = u(x)$ . It follows that  $u_{tt} = 0$ , whence the differential equation characterizing the problem of the equilibrium of the stretched string is

$$(2) \quad S u_{xx} = f(x) , \quad (0 \leq x \leq \ell) .$$

Equations (1) and (2) in themselves are formulations of general laws describing entire classes of phenomena. We are not so much interested in general considerations, however, as we are in special motions. To handle individual cases it is necessary to solve the differential equations supplemented by additional conditions. These conditions take either or both of two different forms, i.e., boundary conditions and initial conditions. The boundary conditions express, independently of time, the nature of the solution on the boundary of the domain in which the





differential equation is defined. In the case under consideration, the stretched string, the boundary consists of the two end-points  $x = 0$  and  $x = \ell$ . If the string is fastened at these points, the boundary conditions read

$$(3) \quad u(0, t) = u(\ell, t) = 0$$

for all values of  $t \geq 0$ .

For both problems of motion and problems of equilibrium, boundary conditions are necessary to obtain a specific solution. Initial conditions, on the other hand, appear only in problems of motion. They express the initial state of the motion, i.e., they specify the deflection and velocity of every point of the string at the instant the motion begins, say, for  $t = 0$ . For the string the initial state is

$$(4) \quad \begin{cases} u(x, 0) = \varphi(x) \\ u_t(x, 0) = \psi(x) \end{cases} \quad (0 \leq x \leq \ell)$$

where  $\varphi$  and  $\psi$  are any prescribed functions.

The question of equilibrium of the string need not be taken up in detail here, since a glance at equation (2) and the boundary conditions (3) reveals that  $u(x)$  is found by integrating  $f(x)$  twice and then making the two constants of integration conform to the boundary conditions. Thus the solution is given by

$$(5) \quad u(x) = \frac{1}{8\ell} [F(x) - xF(\ell)] \quad \text{where} \quad F(x) = \int_0^x \int_0^\xi f(\eta) d\eta d\xi.$$

Exercise: Solve the equilibrium problem for the string with the boundary conditions (3) for the case where

a)  $f(x) = x(x - \ell)$

b)  $f(x) = \sin^2 \frac{x\pi}{\ell}$

c)  $f(x) = 0$  except in the interval of length  $\epsilon$  about the point  $x = \ell/2$  where  $f(x) = 1/\epsilon$ . After solving this boundary value problem, let  $\epsilon$  tend to zero. This will yield the effect of a unit force concentrated at the point  $x = \ell/2$ . Repeat the same problem with the concentrated force acting at any point  $x = \xi$  of the interval  $0 \leq x \leq \ell$  other than the



center. Later we shall see that this process leads to the so-called Green's function for the differential equation (2). By means of this function the solutions of general boundary value problems may be given in a very symmetric form.

In the more interesting case of a vibrating string, the problem may be interpreted geometrically as follows: In the

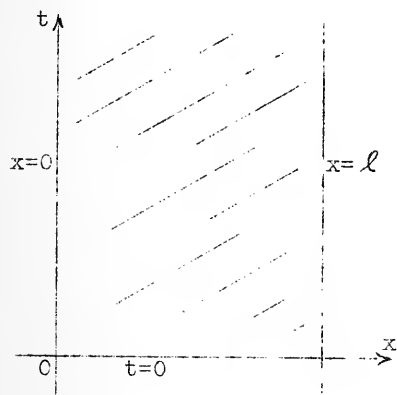


Fig. 6

$x, t$ -plane consider a parallel half-infinite strip bounded by the lines  $x = 0$ ,  $x = l$ , and  $t = 0$ , as in figure 6. Finding the motion of a string with boundary state (3) and initial state (4) becomes the problem of finding a function  $u(x, t)$ , defined in the strip, which satisfies equation (1), vanishes on the lines  $x = 0$  and  $x = l$ , and for which  $u$  and  $u_t$  are prescribed on the line  $t = 0$ . This is the so-called mixed boundary-initial value problem of the differential equation (1).

## 2. The Elastic Membrane

An elastic membrane is defined simply as a thin elastic body which resists stretching but offers no resistance to bending. We assume, too, that the increase in elastic tension is proportional to the increase in area of the membrane. We consider a membrane stretched over a domain  $B$  of the  $x, y$ -plane. The boundary curve  $C$  of  $B$  we assume to be continuous with a piece-wise continuously turning tangent. We will suppose the motion of the membrane to be such that each point moves in a line perpendicular to the  $x, y$ -plane. Denoting the horizontal tension by  $S$ , this implies that  $S$  is a constant over the whole membrane. The normal deflection of any point of the membrane from the  $x, y$ -plane we denote by  $u$ . The quantity  $u$  is, for the equilibrium problem of the membrane, a function  $u = u(x, y)$  defined over the domain  $B$ , while for the problem of vibratory



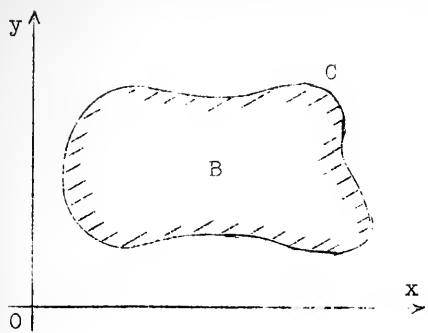


Fig. 7

motion  $u$  is a function of time as well as of  $x$  and  $y$ :  $u = u(x, y, t)$ . As in the case of the string, the mass density  $m$  will be considered constant and the external force density represented by  $f(x, y)$  or  $f(x, y, t)$  according as the problem is one of equilibrium or of motion. In general, the membrane is the two dimensional analog of the elastic string.

To derive the differential equation for the general motion of a membrane we proceed as follows: Let  $G'$  be a small portion of the membrane in a deformed position, and  $G$  the projection of  $G'$  on  $B$ . Let  $K'$  be the boundary of  $G'$  and  $K$  that of  $G$ . The horizontal component  $S$  of the tension is constant. The effective tension, i.e., the vertical component, is given by

$$S \frac{\partial u}{\partial n},$$

as in the case of the string, where  $n$  is the outward unit normal to the curve  $K$ . If  $s$  represents the arc length of  $K$  as measured from some fixed point of  $K$ , then the total elastic force exerted on the element  $G'$  of the membrane is

$$S \int_K \frac{\partial u}{\partial n} ds.$$

We now make use of the Gauss integral theorem\* which transforms the integral over the boundary curve  $K$  into a double integral over the domain  $G$ , namely,

$$S \int_K \frac{\partial u}{\partial n} ds = S \int_G \nabla^2 u dx dy$$

where  $\nabla^2 u = u_{xx} + u_{yy}$  (often denoted by  $\Delta u$ ).

---

\* Courant: Diff. and Int. Calculus, Vol. II, pages 359 ff.



On the other hand we have the inertial force

$$\iint_G m u_{tt} dx dy$$

and the external impressed force

$$\iint_G f(x,y,t) dx dy .$$

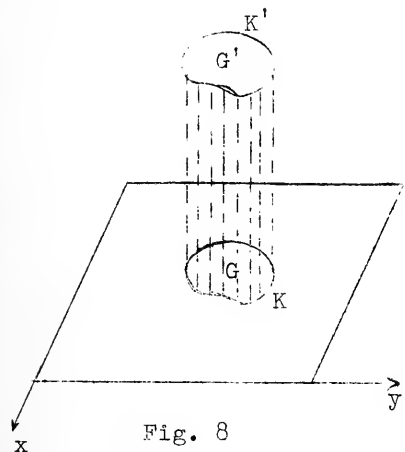


Fig. 8

The elastic force due to the tension in the membrane must be in equilibrium with two latter forces, whence

$$\iint_G (S \nabla^2 u - m u_{tt} - f(x,y,t)) dx dy = 0.$$

Dividing this integral by the area of  $G$  and then letting  $G$  shrink to a point, it follows that the integrand vanishes at every point of  $B$ . This yields the equation of motion for the membrane, i.e.,

$$(6) \quad S \nabla^2 u - m u_{tt} = f(x,y,t) .$$

The problem of equilibrium for the membrane is not a trivial one as was that for the string. This we see if we assume  $f$  and  $u$  independent of  $t$ . Equation (6) then becomes

$$(7) \quad S \nabla^2 u = f(x,y) ,$$

a partial differential equation. The equilibrium equation (2) for the string, on the other hand, was simply an ordinary differential equation.

In order to characterize a specific state of equilibrium or motion of a membrane we must supplement the differential equation by boundary or initial conditions or both. If the membrane is to be fastened all along the boundary  $C$  of the domain  $B$ , then the boundary condition is





$$(8) \quad u = 0 \quad \text{along } C \text{ for all } t \geq 0 .$$

The initial conditions are given by prescribing an initial state for the motion of all points of B, thus

$$(9) \quad \begin{cases} u(x,y,0) = \varphi(x,y) \\ u_t(x,y,0) = \psi(x,y) \end{cases}$$

where  $\varphi$  and  $\psi$  are functions defined over B. For a specific motion we then have a mixed boundary-initial value problem given by the differential equation (6) together with the added conditions (8) and (9). For a case of equilibrium a pure boundary value problem occurs, i.e., equation (7) together with condition (8).

### 3. The Question of Small Deviations

As in Chapter One, it should be emphasized here that the derivation and use of our equations are based on the assumption that all deviations and forces are "small". In consequence, high powers of these quantities can be neglected in comparison with low powers. Although, in the treatment of the equations which we have been discussing, this assumption of "smallness" is usually discarded or forgotten, it should always be kept in mind. Mathematically, however, we are justified in not mentioning the assumption continually. This we see from the following considerations.

Let  $\varepsilon$  be so small a quantity that  $\varepsilon^2$  and higher powers may be neglected, and let us assume that all functions which occur have the form of a product of  $\varepsilon$  by another function which need not be "small" but, on the other hand, is never infinite. We may then derive our differential equations of motion and equilibrium and, in the end, discard the  $\varepsilon$  (it will occur homogeneously in the equations). The resulting equations will be as we have found them to be. In effect then, we may just as well proceed to solve our equations without stipulating "smallness" on the part of any quantities. However, after arriving at the solution, we must consider the physical solution to be  $\varepsilon$  times the mathematical solution.



Bearing in mind these remarks, it will be perfectly legitimate to consider other boundary conditions along  $C$  than that given by (8). For example, we might prescribe  $u = g(s)$  on  $C$ , where  $g$  is a given function of the arc-length  $s$  along  $C$ . Subsequently we shall encounter still different types of boundary conditions.

## Part II. Solution of Equilibrium Problems for the Membrane

### 1. General Formulation

Before discussing the vibrations of strings and membranes and the allied problems of natural frequencies and modes, we shall investigate the question of equilibrium for membranes. Using the notations of Part I the equilibrium problem for the membrane is formulated mathematically as follows.

Let the membrane be stretched over a domain  $B$  in the  $x,y$ -plane with a continuous and piece-wise smooth boundary curve  $C$ . The problem is to find a solution  $u = u(x,y)$  in  $B$  of

$$(1) \quad S \nabla^2 u = f(x,y)$$

satisfying the boundary condition

$$(2) \quad u = g(s) \quad \text{along } C.$$

Here  $S$  is a constant and the functions  $f(x,y)$  and  $g(s)$  are given functions which we assume to be at least piece-wise continuous. The quantity  $s$  is the arc-length measured along  $C$ .

First of all we simplify the problem by supposing that no external force is present, that is,  $f = 0$ . The partial differential equation (1) then becomes

$$(3) \quad \nabla^2 u = 0.$$

This is the famous Laplace equation for harmonic functions.

Later we shall see that this problem and the more general problem involving equation (1), which, incidentally, is called the Poisson equation, are mathematically equivalent. For the present, however, we shall confine ourselves to the formally simpler problem of Laplace where the non-homogeneity enters only in the boundary condition (2).



This boundary value problem corresponds physically to finding the equilibrium position of an elastic membrane B whose non-planar character arises from a prescribed deformation at its boundary C. Students who are familiar with the theory of elasticity know that such problems arise in the theory of torsion. Boundary value problems of this type also play an important role in other fields. For example, the equilibrium state of heat transfer in a plane conductor is also governed by the Laplace equation, where, in this case,  $u$  denotes the temperature. If a flat conductor with the shape of the domain B has its boundary C kept at a certain fixed temperature given by  $g(s)$ , then the solution of the boundary value problem represents the steady state of the temperature of B which will have developed after the lapse of some time.

We shall begin the systematic discussion and solution of the boundary value problem for the Laplace equation with two very significant examples.

## 2. The Boundary Value Problem for the Circular Membrane

We consider first the boundary value problem

$$(2) \quad u = g(s) \quad \text{on } C$$

$$(3) \quad \nabla^2 u = 0 \quad \text{in } B$$

for the case where the curve C is a circle and the domain B its interior. For simplicity let C be a circle of radius with center at the origin.

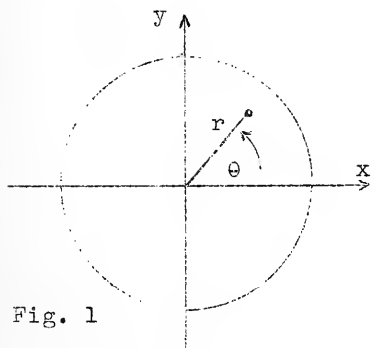


Fig. 1

Introducing polar coordinates  $r$  and  $\theta$  by the transformation formulas

$$(4) \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases},$$

the domain B is given analytically by the inequality  $r < 1$  and the boundary C by  $r = 1$ . Furthermore, since  $s = r\theta$ , the boundary



condition  $u = g(s)$  may be written as a function of  $\theta$ , e.g.,  $u = g(\theta)$ . This function is, of course, periodic with the period  $2\pi$ .

Our method of attack for this problem will be to find an infinite set of particular solutions to the differential equation (3) and then to superimpose them to form a solution satisfying the boundary condition (2).

The problem assumes a more symmetric form if we transform the Laplace equation from its rectangular form to the equivalent equation in polar form. Using equations (4) we obtain

$$(5) \quad \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad .$$

To obtain particular solutions of (5) we make the "product trial"

$$(6) \quad u(r, \theta) = h(\theta) k(r) \quad .$$

Substituting (6) into (5) we get

$$\frac{r^2 [k''(r) + \frac{1}{r} k'(r)]}{k(r)} = - \frac{h''(\theta)}{h(\theta)} \quad .$$

Now, since the right side of this equation depends only on  $\theta$  and the left only on  $r$ , they can be equal only if they are equal to one and the same constant  $c$ . Setting each side equal to  $c$  we arrive at two ordinary differential equations,

$$(7) \quad h''(\theta) + c h(\theta) = 0$$

$$(7') \quad r^2 k''(r) + r k'(r) - c k(r) = 0 \quad ,$$

for the functions  $h(\theta)$  and  $k(r)$ .

Solutions of (7) are readily obtained in the form  $e^{\pm \theta \sqrt{-c}}$  where  $\sqrt{-c}$  is, of course, real or imaginary according as  $c$  is negative or positive. Since  $u(r, \theta)$  must be periodic with period  $2\pi$  in  $\theta$ , it follows that  $h(\theta)$  must be also, whence not only must  $c$  be positive but, also,  $\sqrt{c}$  must be an integer. Hence we set  $c = n^2$  where  $n$  is an integer. Consequently, we have solutions of (7) of the form





$$h(\theta) = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}.$$

Inserting  $c = n^2$  in (7') we have

$$(8) \quad r^2 k''(r) + rk'(r) - n^2 k(r) = 0.$$

This equation appears to be much more difficult to solve than (7) since the coefficients here are non-constant. However, the function

$$k(r) = r^n$$

is easily seen to be a solution. Hence we have found infinitely many particular solutions of the Laplace equation (5). They are of the form

$$(9) \quad \begin{cases} r^n \cos n\theta \\ r^n \sin n\theta \end{cases}$$

where  $n$  is any non-negative integer.

Before proceeding with the solution we remark that there is a much more elegant method for obtaining the above particular solutions of the Laplace equation. Consider the Laplace operator  $\nabla^2 u = u_{xx} + u_{yy}$  and the complex number  $z = x + iy$ . It is obvious that  $z^n = (x + iy)^n$  is a solution of  $\nabla^2 u = 0$  since  $u_{xx} = n(n-1)z^{n-2}$  and  $u_{yy} = i^2 n(n-1)z^{n-2} = -n(n-1)z^{n-2}$ . If we introduce polar coordinates by the transformations (4), we find  $z^n = r^n(\cos \theta + i \sin \theta)$  or, by De Moivre's theorem,  $z^n = r^n \cos n\theta + i r^n \sin n\theta$ . However, since  $u$  and  $\nabla^2 u$  are real, both real and imaginary parts of  $z^n$  separately must be solutions of  $\nabla^2 u = 0$ . Hence we again obtain the particular solutions  $r^n \cos n\theta$  and  $r^n \sin n\theta$ .

Returning now to the solutions (9) we attempt to satisfy the boundary condition  $u(1, \theta) = g(\theta)$  by the following superposition of these solutions, which, of course, will again be a solution of (5), namely,

$$(10) \quad u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$



where  $a_0$ ,  $a_n$ , and  $b_n$  are undetermined constants. In order to satisfy the boundary condition we must have

$$(11) \quad u(1, \theta) = g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad .$$

If we assume that  $g(\theta)$  may be developed into a Fourier series, then, in order that (11) be satisfied, the constants  $a_n$  and  $b_n$  must be the Fourier coefficients of  $g(\theta)$ . Thus, equation (10), which is a solution of the Laplace equation satisfies the boundary condition  $u(1, \theta) = g(\theta)$  if

$$(12) \quad \begin{cases} a_n = \frac{1}{\pi} \int_0^{2\pi} g(\varphi) \cos n\varphi d\varphi & (n = 0, 1, 2, \dots) \\ b_n = \frac{1}{\pi} \int_0^{2\pi} g(\varphi) \sin n\varphi d\varphi & (n = 1, 2, \dots) \end{cases} \quad .$$

Remark: It is always desirable to denote integration variables by different letters than those employed in the discussion and, in general, to avoid the use of the same letter for different quantities in the same or related formulas.

We have solved the boundary value problem for the unit circle. For a circle of radius  $R$  the solution is

$$(13) \quad u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (a_n \cos n\theta + b_n \sin n\theta)$$

as may be easily verified.

It is necessary to make a few remarks concerning the convergence of the Fourier series for  $g(\theta)$ . We mentioned in section 1 that  $g(\theta)$  was to be at least piece-wise continuous,

i.e., continuous except for a finite number of jump discontinuities (see figure 2).

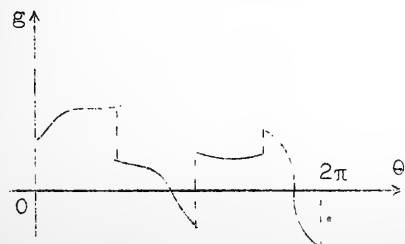


Fig. 2

Our solutions (10) and (13), however, appear to require that  $g(\theta)$  be representable by a uniformly convergent Fourier series. Piece-wise continuity, or even continuity,



does not insure that such a representation is possible. If  $g(\theta)$  is assumed to be not only piece-wise continuous but also piece-wise smooth (i.e., possessing a piece-wise continuous derivative), then a Fourier series representation is possible. It can be shown, however, that solutions (10) and (13) are valid if  $g(\theta)$  is assumed to be merely piece-wise continuous or continuous.

Suppose  $g(\theta)$  to be rarely continuous and nothing more throughout the interval  $0 \leq \theta \leq 2\pi$ . It is then possible to approximate  $g(\theta)$  uniformly by a sequence of polynomials  $P_n(\theta)$  in  $\cos \theta$  and  $\sin \theta$ , i.e.,

$$P_n(\theta) = \frac{\alpha_0^{(n)}}{2} + \sum_{\nu=1}^n (\alpha_\nu^{(n)} \cos \nu \theta + \beta_\nu^{(n)} \sin \nu \theta)$$

where

$$\alpha_\nu^{(n)} = \frac{1}{\pi} \int_0^{2\pi} P_n(\varphi) \cos \nu \varphi d\varphi \quad (\nu = 0, 1, 2, \dots, n)$$

and

$$\beta_\nu^{(n)} = \frac{1}{\pi} \int_0^{2\pi} P_n(\varphi) \sin \nu \varphi d\varphi \quad (\nu = 1, 2, \dots, n)^*.$$

The function

$$u_n(r, \theta) = \frac{\alpha_0^{(n)}}{2} + \sum_{\nu=1}^n r^\nu (\alpha_\nu^{(n)} \cos \nu \theta + \beta_\nu^{(n)} \sin \nu \theta)$$

then solves the boundary value problem for the circle with the boundary condition  $u_n(1, \theta) = P_n(\theta)$ . If we now let  $n \rightarrow \infty$ , the function  $u_n(r, \theta)$  will tend uniformly to a function  $u(r, \theta)$  which, according to a well-known theorem (Harnack's theorem) of potential theory, satisfies the equation  $\nabla^2 u = 0$  in the unit circle and takes on boundary values given by the limit of  $P_n(\theta)$  as  $n \rightarrow \infty$  which is  $g(\theta)$ . The limit function  $u(r, \theta)$  is given by (10) where

---

\* Students familiar with the theory of Fourier series will recognize this as the Fejer approximation theorem. It may be proved from the Weierstrass approximation theorem which states that any function continuous in a closed interval may be uniformly approximated to any degree of accuracy by a polynomial.



$$a_n = \lim_{m \rightarrow \infty} \alpha_n^{(m)} = \frac{1}{\pi} \int_0^{2\pi} g(\varphi) \cos n\varphi d\varphi \quad (n = 0, 1, 2, \dots)$$

$$b_n = \lim_{m \rightarrow \infty} \beta_n^{(m)} = \frac{1}{\pi} \int_0^{2\pi} g(\varphi) \sin n\varphi d\varphi \quad (n = 1, 2, \dots).$$

Thus (10) is a valid solution under the assumption that  $g(\theta)$  is merely continuous.

If  $g(\theta)$  is piece-wise continuous we mention, without proof, that in each interval of continuity the solution is given by (10) and at the points of discontinuity,  $\theta_\nu$ : ( $\nu = 1, 2, \dots, m$ ), the expression (10), for  $r = 1$ , yields a value of  $u(1, \theta)$  which lies between  $g(\theta_\nu + 0)$  and  $g(\theta_\nu - 0)$ , i.e., between the right-hand and left-hand limits of  $g(\theta)$  as  $\theta \rightarrow \theta_\nu$ . Hence the Laplace boundary value problem for the unit circle with a piece-wise continuous boundary condition is given by (10) and the coefficients  $a_n$  and  $b_n$  by (12). Also, (13) yields the solution for a circle of radius  $R$ .

Exercise: Solve the boundary value problem for the unit circle where

$$a) \quad g(\theta) = \sin^3 \theta, \quad 0 \leq \theta \leq 2\pi$$

$$b) \quad g(\theta) = \theta(\theta - 2\pi), \quad 0 \leq \theta \leq 2\pi$$

$$c) \quad g(\theta) = \begin{cases} 1 & \text{for } 0 \leq \theta \leq 2\pi/3 \\ -1 & \text{for } 2\pi/3 \leq \theta \leq 4\pi/3 \\ 0 & \text{for } 4\pi/3 \leq \theta \leq 2\pi \end{cases}$$

### 3. Integral Representation of the Solution for the Circle. Poisson's Integral

We give here another form of the solution (13) of the boundary value problem for a circle of radius  $R$ . This new form, the so-called Poisson's Integral, is more of mathematical interest than it is of practical use. We obtain it as follows.

In equation (13) insert in place of  $a_n$  and  $b_n$  their values as given by (12). Thus





$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \left[ \cos n\theta \int_0^{2\pi} g(\varphi) \cos n\varphi d\varphi + \sin n\theta \int_0^{2\pi} g(\varphi) \sin n\varphi d\varphi \right].$$

Employing the trigonometric formula

$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ , we get

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \int_0^{2\pi} g(\varphi) \cos n(\theta - \varphi) d\varphi.$$

We now interchange integration and summation (which is possible because of the uniform convergence of  $u$  for  $r < R$ ) and obtain

$$u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n \cos n(\theta - \varphi) \right\} g(\varphi) d\varphi.$$

Writing  $2\cos \alpha = e^{i\alpha} + e^{-i\alpha}$ , the last expression becomes

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) \left\{ 1 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (e^{in(\theta - \varphi)} + e^{-in(\theta - \varphi)}) \right\} d\varphi.$$

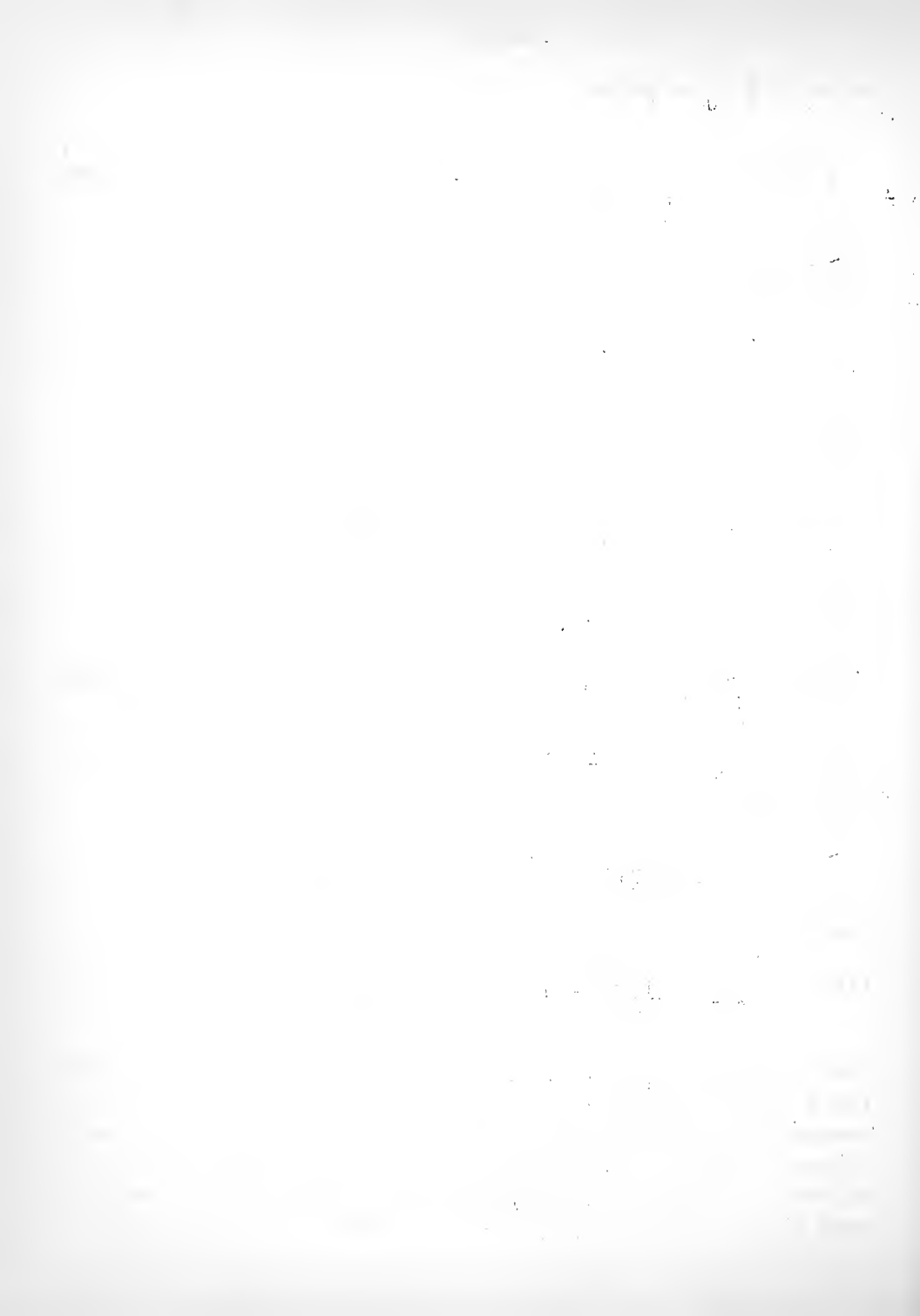
Now, the sums  $\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n e^{\pm in\alpha}$  are infinite geometric series which can be summed for  $r < R$ , yielding

$$\sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n e^{\pm in\alpha} = -1 + \frac{R}{R - re^{\pm i\alpha}}.$$

Hence, we find

$$(14) \quad u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} d\varphi.$$

This is Poisson's Integral. It solves the boundary value problem for a circle of radius  $R$  in terms of an explicit expression involving the boundary value  $g(\theta)$ . We notice that if  $r = R$  the integral exhibits a singularity in that the numerator vanishes, as does the denominator for  $\varphi = \theta$ . We remark, without proof, that it is precisely this singularity that makes the value of the integral  $g(\theta)$  for  $r = R$ .



#### 4. The Rectangular Membrane

As a second example of the equilibrium problem for a membrane, we consider the case where the domain  $B$  is a rectangle.

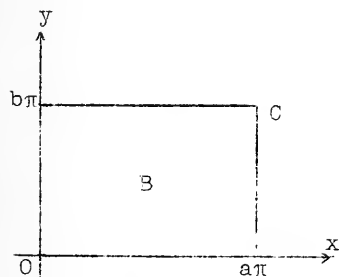


Fig. 3

For the sake of simplicity we let the rectangle have sides of lengths  $a\pi$  and  $b\pi$ . Also, assume that the rectangle is located as in the figure with two of its sides falling along the axes. The boundary curve  $C$  consists of segments of four lines  $x = 0$ ,  $x = a\pi$ ,  $y = 0$ ,  $y = b\pi$ .

The problem is greatly simplified when we recognize that it is necessary to assume boundary

values different from zero on one side of the rectangle only. For, by the addition of the solutions for four problems of this type, one for each side, we obtain the solution to the more general problem.

Because of this we suppose that the prescribed boundary function  $g(x,y)$  is identically zero on the sides  $x = 0$ ,  $x = a\pi$ ,  $y = 0$ , and equal to a given function  $\varphi(x)$  on  $y = b\pi$ , where  $0 \leq x \leq a\pi$ .

We proceed to a solution of our problem by making a "product trial"

$$(15) \quad u(x,y) = k(x) h(y) \quad .$$

Substituting this into the Laplace equation (3) we get

$$\frac{k''(x)}{k(x)} = - \frac{h''(y)}{h(y)}$$

whence we obtain two ordinary differential equations, one for  $k(x)$  and one for  $h(y)$ , namely,

$$(16) \quad k''(x) - ck(x) = 0$$

$$(17) \quad h''(y) + ch(y) = 0$$



where  $c$  is an arbitrary constant. Since, by the boundary conditions,  $u(0, y) = u(a, y) = 0$ , it follows that the desired solution of (16) must satisfy  $k(0) = k(a) = 0$ . Hence  $c$  must be negative and  $\sqrt{-c} = n/a$ , ( $n = 1, 2, \dots$ ), since

$$k(x) = \sin \frac{n}{a} x \quad (n = 1, 2, \dots)$$

are the only solutions of (16) satisfying these conditions. Inserting these values of  $c$  in (17) we find a solution of (17) in the form

$$h(y) = \alpha_1 e^{\frac{n}{a} y} + \alpha_2 e^{-\frac{n}{a} y}$$

where the  $\alpha$ 's are arbitrary constants. In order to satisfy the boundary condition  $u(x, 0) = 0$  we must have  $h(0) = 0$ , i.e.,  $\alpha_2 = -\alpha_1$ . Since

$$\frac{1}{2}(e^z - e^{-z}) = \sinh z$$

( $\sinh z =$  hyperbolic sine of  $z$ ), we may write the required solutions of (17) in the form

$$h(y) = \sinh \frac{n}{a} y \quad (n = 1, 2, \dots) .$$

Hence we have an infinity of particular solutions

$$(18) \quad u(x, y) = \sin \frac{n}{a} x \sinh \frac{n}{a} y$$

of the Laplace equation (3) which satisfy the required boundary conditions on three of the four sides of the rectangle. By superimposing these solutions we obtain another solution of (3) which also satisfies the same boundary conditions, i.e.,

$$(19) \quad u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n}{a} y \sin \frac{n}{a} x$$

where the  $A_n$  are arbitrary constants. The  $A_n$  are now to be determined by requiring that  $u(x, b)$  coincide with the prescribed boundary function  $\varphi(x)$ . That is,

$$(20) \quad u(x, b) = \varphi(x) = \sum_{n=1}^{\infty} A_n \sinh \frac{nb}{a} \sin \frac{n}{a} x$$

... ..  
... ..  
... ..

... ..  
... ..

... ..  
... ..

... ..  
... ..

... ..  
... ..

... ..  
... ..  
... ..

... ..

... ..  
... ..  
... ..

... ..

(20')

$$\varphi(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n}{a} x$$

where

$$B_n = A_n \sinh \frac{n b \pi}{a} .$$

The function  $\varphi(x)$  has been defined in the interval  $0 \leq x \leq a\pi$ . We now extend it symmetrically to the interval  $-a\pi \leq x \leq 0$  and then periodically along the  $x$ -axis with period  $2a\pi$ . The symmetric extension to the interval  $-a\pi \leq x \leq 0$  may be accomplished in two ways.  $\varphi(x)$  may be extended as an even

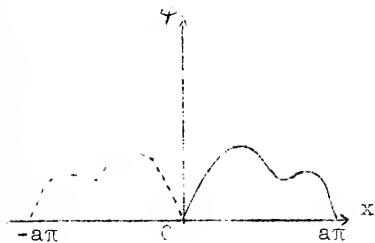


Fig. 4

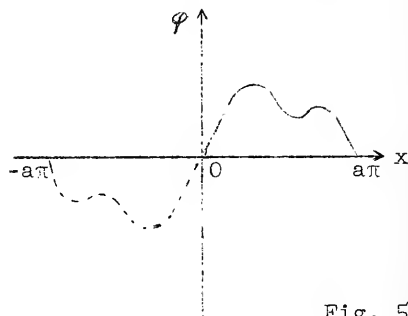


Fig. 5

function\* (figure 4), or as an odd function (figure 5). If we calculate the Fourier coefficients  $a_n$  and  $b_n$  of  $\varphi(x)$ , which we assume to be piece-wise continuous, we get

$$\begin{cases} a_n = \frac{1}{a\pi} \int_{-a\pi}^{a\pi} \varphi(\xi) \cos \frac{n}{a} \xi d\xi & (n = 0, 1, 2, \dots) \\ b_n = \frac{1}{a\pi} \int_{-a\pi}^{a\pi} \varphi(\xi) \sin \frac{n}{a} \xi d\xi & (n = 1, 2, \dots) \end{cases} .$$

If  $\varphi(x)$  is to be even, then  $\varphi(z)\cos z$  must be even and  $\varphi(z)\sin z$  odd. Hence

$$\begin{cases} a_n = \frac{2}{a\pi} \int_0^{a\pi} \varphi(\xi) \cos \frac{n}{a} \xi d\xi & (n = 0, 1, 2, \dots) \\ b_n = 0 \end{cases} .$$

\*A function  $\varphi(x)$  is said to be even if  $\varphi(x) = \varphi(-x)$  and odd if  $\varphi(x) = -\varphi(-x)$ .





whence  $\varphi(x)$  possesses a pure cosine expansion. On the other hand, if  $\varphi(x)$  is to be odd, then  $\varphi(z)\cos z$  is odd and  $\varphi(z)\sin z$  is even. Hence

$$\begin{cases} a_n = 0 \\ b_n = \frac{2}{a\pi} \int_0^{a\pi} \varphi(\xi) \sin \frac{n}{a} \xi d\xi \end{cases}$$

whence  $\varphi(x)$  possesses a pure sine expansion.

Since (20') gives  $\varphi(x)$  as a sine series we choose to extend  $\varphi(x)$  as an odd function. Developing it into a Fourier series of sine terms alone enables us to evaluate the coefficients  $B_n$  of (20'), i.e.,

$$(21) \quad B_n = A_n \sinh \frac{nb\pi}{a} = \frac{2}{a\pi} \int_0^{a\pi} \varphi(\xi) \sin \frac{n}{a} \xi d\xi.$$

Hence the final solution of the boundary value problem is

$$(22) \quad u(x,y) = \sum_{n=1}^{\infty} B_n \frac{\sinh \frac{n}{a} y}{\sinh \frac{nb\pi}{a}} \sin \frac{n}{a} x$$

where  $B_n$  is given by (21).

The discussion of the last section concerning the validity of the solution of the boundary value problem when the boundary function is merely continuous or piece-wise continuous holds in our present case also. In this case, the solution at the points of discontinuity tends to a value which lies between the right- and left-hand limits of  $\varphi(x)$  as  $x$  approaches a point of discontinuity. For this reason we are justified in considering boundary functions which do not vanish at the points  $x = 0$ ,  $y = b\pi$ , or  $x = a\pi$ ,  $y = b\pi$ , or both.

For us the important question concerning the representation of solutions of various problems by infinite series is the practicality of such a solution. That is, we are interested in a solution that is suitable for a numerical calculation. Mathematically, this means that the infinite series involved must converge rapidly enough that numerical results may be obtained with sufficient accuracy by considering only the first few terms of the series.



This is really the case with the series (22), For example, in the neighborhood of the side  $y = b\pi$ , where the boundary function  $\varphi(x)$  subsists,  $u(x,y)$  approaches  $\varphi(x)$  exponentially (which, of course, means very rapidly), since

$$\frac{\sinh \frac{n}{a} y}{\sinh \frac{n b \pi}{a}} = \frac{e^{\frac{n}{a} y} - e^{-\frac{n}{a} y}}{e^{\frac{n b \pi}{a}} - e^{-\frac{n b \pi}{a}}}$$

and the dominating factor in the denominator is  $e^{\frac{n b \pi}{a}}$ . Thus the quotient is approximately

$$\frac{e^{\frac{n}{a} y}}{e^{\frac{n b \pi}{a}}} = e^{-\frac{n}{a} y}.$$

As  $y \rightarrow b\pi$ , the first term tends rapidly to 1 while the second becomes very small exponentially. Likewise, in the neighborhood of the other three sides, each term of the series approaches zero as we approach these sides from the interior. This, also, points to a very rapid convergence in the neighborhood of the boundary of the membrane.

Exercise: Solve the boundary value problem for the rectangular membrane of this section for the case where

a)  $\varphi(x) = x(a\pi - x)$  on  $y = b\pi$ , and zero on the other three sides.

b)  $\varphi(y) = \sin b\pi y$  on  $x = 0$ ,  $\varphi(y) = \sin^2 b\pi y$  on  $x = a\pi$ , and zero on the other two sides.

c)  $\varphi(x) = \begin{cases} x & \text{for } 0 \leq x \leq a\pi/2 \\ a\pi - x & \text{for } a\pi/2 \leq x \leq a\pi \end{cases}$  on  $y = b\pi$

and zero on the other three sides. Calculate the approximate numerical value of the solution at the center of the membrane. Assume, for simplicity, that the membrane is a square of side  $\pi$ , i.e.,  $a = b = 1$ .



### Part III. Method of Finite Differences

The circle and the rectangle are the only domains for which a complete solution of the boundary value problem for the Laplace equation in the plane can be given in terms of elementary functions. It is important, therefore, in view of the great variety of applications, that some general methods be developed to handle the boundary value problem for general types of domains.

One of the most powerful and practical of such general methods is the "method of finite differences". The fundamental idea of this method is to replace the differential equation by a "difference equation" (an equation involving finite differences), thereby reducing the problem to a simple system of linear algebraic equations in a finite number of unknowns. The new problem may then be solved in various ways, making possible the formulation of a routine procedure.

#### 1. The Laplace Difference Equation

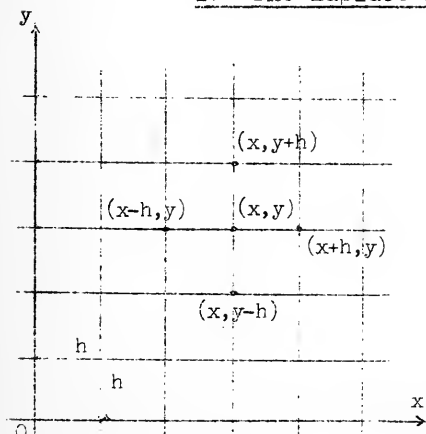


Fig. 1

We begin by covering the  $x, y$ -plane with a quadratic mesh consisting of squares of side  $h$ . To do this we draw in the plane the two sets of parallel lines

$$(1) \begin{cases} x = mh & (m = 0, 1, 2, \dots) \\ y = nh & (n = 0, 1, 2, \dots) \end{cases} .$$

These two families of lines intersect in points which we call the net, or lattice, points of the mesh.

Now, instead of considering functions  $u(x, y)$ ,  $f(x, y)$ , etc., of the continuous variables  $x$  and  $y$ , we consider functions which are defined only at the lattice points of the above mesh. That is, the functions are to be defined solely for the arguments  $x = mh$ ,  $y = nh$ , where  $h$  is some fixed number. In any bounded domain only a finite number of lattice points will be present and hence each function will take on only a finite number of



values. It is impossible to speak of the derivatives of such functions. Instead we define what we call the "difference quotients" of these discrete valued functions.

Let  $u(x,y)$  be a function defined at the lattice points of the  $x,y$ -plane. Then the forward difference quotient of  $u$  with respect to  $x$  at a lattice point  $(x,y)$  is defined to be

$$(2) \quad u_x(x,y) = \frac{u(x+h,y) - u(x,y)}{h}$$

and the backward difference quotient with respect to  $x$ ,

$$(3) \quad u_{\bar{x}}(x,y) = \frac{u(x,y) - u(x-h,y)}{h} .$$

In general, these two difference quotients are not equal. The corresponding difference quotients with respect to  $y$  are

$$(2') \quad u_y(x,y) = \frac{u(x,y+h) - u(x,y)}{h}$$

$$(3') \quad u_{\bar{y}}(x,y) = \frac{u(x,y) - u(x,y-h)}{h} .$$

[Remark: We have used the notation  $u_x$  for the forward difference quotient of  $u$  with respect to  $x$ . This notation should not be confused with our previous use of  $u_x$  to denote the partial derivative of  $u$  with respect to  $x$ . Although the same notation will be used for the two different quantities it will be clear from the context as to what is meant.]

In a manner similar to the above we may define the second difference quotients of a function, i.e., the difference quotients of the first difference quotients. The forward second difference quotient is given by

$$u_{xx}(x,y) = \frac{u_x(x+h,y) - u_x(x,y)}{h} .$$

Substituting for  $u_x(x+h,y)$  and  $u_x(x,y)$  their values as given by (3), we have

$$(4) \quad u_{\bar{x}x}(x,y) = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} .$$





Also

$$\begin{aligned} u_{xx}(x,y) &= \frac{u_x(x,y) - u_x(x-h,y)}{h} \\ &= \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} \end{aligned}$$

whence

$$u_{x\bar{x}} = u_{\bar{x}\bar{x}} \quad .$$

We could, if we wished, consider the second difference quotients  $u_{xx}$  and  $u_{\bar{x}\bar{x}}$ . However, the use of  $u_{\bar{x}\bar{x}}$  makes for greater symmetry.

In a similar manner, we have

$$(4') \quad u_{yy}(x,y) = \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2} \quad .$$

We now replace the Laplace operator  $\nabla^2$  by the difference operator, which we denote by  $\nabla_h^2$ , to apply to functions defined only at the lattice points (1). Thus

$$(5) \quad \nabla_h^2 u = u_{x\bar{x}} + u_{y\bar{y}}$$

or, by (4) and (4'),

$$\nabla_h^2 u = \frac{1}{h^2} [u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)] \quad .$$

The significance of this operator becomes clear if we consider a net point  $P_0$  and its four neighboring net points  $P_1, P_2, P_3, P_4$ . (Two net points are called neighbors if the distance between them is  $h$ .) Denoting by  $u_j$  the value of  $u$  at  $P_j$ , we have, for the value of  $\nabla_h^2 u$  at  $P_0$ ,

$$(6) \quad \nabla_h^2 u = \frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{h^2} \quad .$$

That is, the value of  $\nabla_h^2 u$  at  $P_0$  is four times the excess of the arithmetic mean of the four neighboring values over  $u_0$ , this means being divided by the area  $h^2$  of the mesh. As a



matter of fact, this is the underlying significance of the Laplace operator even in its original derivative form.

The equation  $\nabla^2 u = 0$  corresponds to the difference equation obtained from (6) by equating the right-hand side to zero, thus

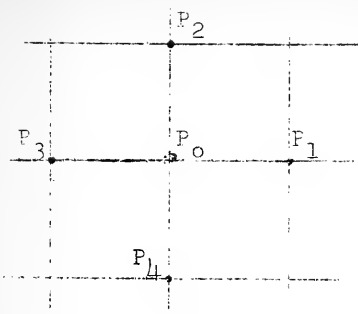


Fig. 2

$$(7) \quad u_0 = \frac{u_1 + u_2 + u_3 + u_4}{4}.$$

Hence, in a quadratic net, the Laplace equation states that the value of  $u$  at a lattice point  $P$  is the arithmetic mean of the values of  $u$  at the four neighbors of  $P$ . Students familiar with potential theory know that even solutions of  $\nabla^2 u = 0$  possess this remarkable mean value property, where, of course, in the latter case, the neighbors of  $P$  are the points of the circumference of a circle about  $P$  as a center.

## 2. Boundary Value Problem in a Net.

To develop the method of finite differences for the equilibrium problem of the membrane in a general domain  $B$ , we proceed as follows.

a) Cover the  $x, y$ -plane with a quadratic net, the lines defining the net being given by (1).

b) Let  $B$  be any bounded domain in the  $x, y$ -plane with a piece-wise smooth boundary  $C$ . The "net domain"  $B_n$  corresponding to the domain  $B$  consists of all the net points which lie in  $B$ . A net point is said to be a boundary

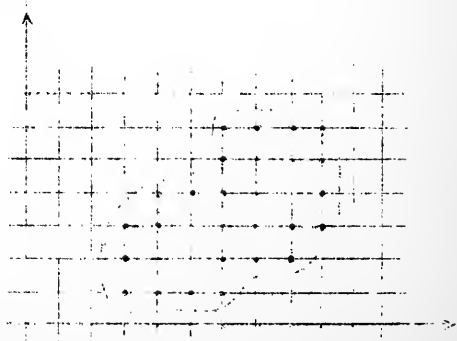


Fig. 3



point of  $B_h$  if not all of its four neighbors are in  $B$ ; if all four neighboring points are in  $B$ , the point is said to be an interior point of  $B_h$ . The boundary  $C_h$  of  $B_h$  is defined as the set of all boundary points of  $B_h$ . A polygonal path, as in figure 3, with sides parallel to the  $x$  and  $y$  axes, passes through or encloses all the points of  $B_h$ .

c) To solve, with any specified degree of accuracy, the boundary value problem of the differential equation  $\nabla^2 u = 0$  for the domain  $B$ , we replace the differential equation by the difference equation

$$(8) \quad \nabla_h^2 u = 0$$

and the domain  $B$  by the corresponding net domain  $B_h$ .

If  $u = g(s)$  is the boundary function prescribed on  $C$ , then our boundary values at the points of  $C_h$  are chosen as follows. If a net point

$P$  of  $C_h$  lies on  $C$ , then the value of  $g(s)$  at  $P$  is taken as the value of  $u$  at  $P$ ; if  $P$  does not lie on  $C$ , we choose as the value of  $u$  at  $P$  the value of  $g(s)$  at a point  $Q$  of  $C$  near to  $P$ . (See figure 4).

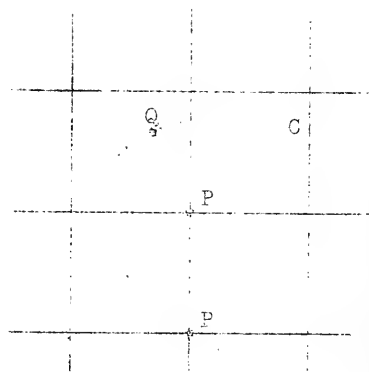


Fig. 4

d) We may now solve the boundary value problem for the net domain by methods which we shall discuss below, obtaining thereby an approximate solution to the original boundary value problem for the Laplace equation.



Fig. 5



We consider net domains such as those shown in figure 5, where at each boundary point a boundary value for  $u$  is prescribed. For example, figure 6 shows a square net domain

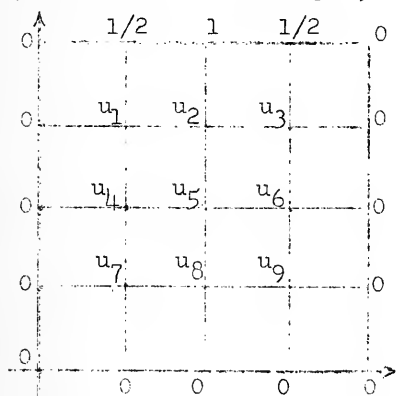


Fig. 6

containing 25 net points. At each of the 16 boundary points a prescribed value of  $u$  is indicated. At each of the 9 interior points the value of  $u$  is to be found.

In general, denote by  $N$  the number of interior points of the net domain  $B_h$ . Using (7) we may set up for each of the interior points the difference equation  $\nabla_h^2 u = 0$ . In each case this is a linear equation involving five

values of  $u$ . Some of these equations contain known quantities, i.e., those for points that are neighbors of boundary points. The other equations are homogeneous. Altogether we obtain a system of  $N$  linear equations in  $N$  unknowns, i.e., the values of  $u$  at the  $N$  interior points. For example, in the case of the square net domain of figure 6 we have, in accordance with the indicated notation,

$$\begin{aligned} u_1 &= \frac{1}{4}(u_4 + u_2 + \frac{1}{2}), & u_4 &= \frac{1}{4}(u_1 + u_5 + u_7), & u_7 &= \frac{1}{4}(u_4 + u_8), \\ u_2 &= \frac{1}{4}(u_5 + u_3 + u_1 + 1), & u_5 &= \frac{1}{4}(u_2 + u_4 + u_6 + u_8), & u_8 &= \frac{1}{4}(u_5 + u_7 + u_9), \\ u_3 &= \frac{1}{4}(u_6 + u_2 + \frac{1}{2}), & u_6 &= \frac{1}{4}(u_3 + u_5 + u_9), & u_9 &= \frac{1}{4}(u_6 + u_8). \end{aligned}$$

The fact that we have  $N$  equations in  $N$  unknowns insures the possibility of obtaining a solution. However, it is of great importance for the actual calculation of the solution that we take into account the particular simplicity of our equations. The following procedure will prove useful in many cases.





Suppose we know the values of  $u$  at points  $P$  of two successive rows of the net.

We can then calculate the values of  $u$  at points  $Q$  of the next row that are included in the longest rectangle which is determined by the points  $P$  and which contains no net points exterior to the domain  $B$ . (See figure 7). Thus, for example,

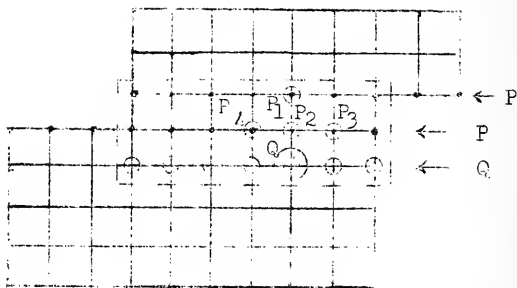


Fig. 7

$$u(Q) = 4u(P_2) - u(P_1) - u(P_3) - u(P_4)$$

is obtained by applying  $\nabla_h^2 u = 0$  at  $P_2$ . This can be done for each  $Q$  except the one at the extreme right. The latter, however, is a boundary point at which  $u$  is prescribed.

As an example of the use of this method consider again the square of figure 6. Because of the symmetry in the boundary values we may expect a corresponding symmetry in the solution, i.e., we assume  $u_1 = u_3 = v$  and  $u_2 = w$ . Then in the next horizontal row we have

$$u_4 = 4v - w - \frac{1}{2}$$

$$u_5 = 4w - 2v - 1$$

$$u_6 = u_4$$

In the next row

$$u_7 = 4u_4 - u_5 - v = 17v - 8w - 1$$

$$u_8 = 4u_5 - u_4 - u_6 - w = 17w - 16v - 3$$

$$u_9 = u_7$$



Using the fact that the boundary values are zero at the bottom of the square, we have

$$4u_7 - u_8 - u_4 = 0$$

$$4u_8 - u_7 - u_9 - u_5 = 0$$

Substituting in these two equations for  $u_4, u_5, u_7, u_8, u_9$ , the values of these quantities in terms of  $v$  and  $w$ , we obtain a system of two linear equations in  $v$  and  $w$  which are easily solved. Immediately, then, the values of  $u_4, \dots, u_9$  can be found by the preceding relations.

This procedure is particularly suited to domains of the type shown in figure 9 which taper off toward the bottom, so that, starting at the top, all values in the succeeding rows may be determined.

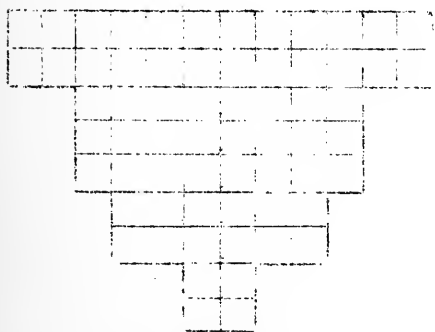


Fig. 9

We begin by assuming for  $u(x,y)$  at the interior net points of  $B_h$  any values whatsoever. It is desirable, however, to make this "first approximation" (which we denote by  $u_1$ ), in such a way that the assumed values lie between the maximum and

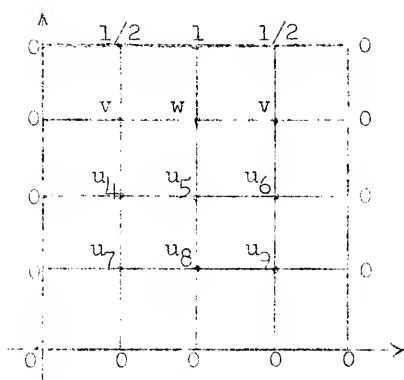


Fig. 8

Often, of course, we have to deal with domains which display no symmetries and for which the boundary conditions are unsymmetric. The shortcut mentioned above cannot be used in such cases. Rather, more general procedures present themselves. These methods consist of processes of repeated manipulations which may be performed mechanically.



minimum of the boundary values\*. We now consider two procedures.

a) Order the interior net points of  $B_h$  in some arbitrary manner,  $P_1, P_2, \dots, P_N$ . Then replace  $u_1(P_1)$  (our assumed "first approximation" at  $P_1$ ) by the arithmetic mean of the assumed values  $u_1$  at the four neighbors of  $P_1$ . Using this value, do the same for  $u_1(P_2)$ . Using the new values at  $P_1$  and  $P_2$ , repeat the process for  $u_1(P_3)$ . Continue this process until the values of  $u_1$  at all  $N$  of the interior points have been "corrected". The corrected values we denote by  $u_2(P)$ . They give us a "second approximation" to the final solution. Again we start out with  $P_1$  and proceed, exactly as before, to determine a "third approximation"  $u_3(P)$ . This process is to be continued as long as notable differences occur between a value and its replaced value. When this is no longer the case, we consider these values to be a good approximation to the actual solution of the boundary value problem.

b) Instead of proceeding as in a), where consecutive replacements were made, we obtain a second approximation immediately by replacing  $u_1(P)$  by the mean value of the first approximation  $u_1(Q)$  at the four neighbors  $Q$ . Thus the second approximation is obtained directly from the first approximation. In the same manner we obtain a third approximation  $u_3(P)$  directly from  $u_2(P)$ , and so on. After a while the values will no longer change noticeably and a satisfactory solution is thus obtained.

A remarkable merit of these two methods is that, in the long run, they tend to correct any numerical errors which may have been made during the replacements. This may be attributed to the fact that we continue to take arithmetic means as long as marked changes occur.

---

\*According to well-known patterns, the maximum and minimum values of  $u(x,y)$  are attained at the boundary. The proof is based on the mean value property (7) of the Laplace difference equation. If the maximum were attained at an interior point  $P$ , then the value of  $u$  at one of the four neighbors, say  $Q$ , would be at least as large as at  $P$ . If this value is larger than that at  $P$  we have a contradiction and, hence, the statement is proved. If the two values are equal, by continuing such a process about  $Q$  we see that  $u$  must be constant in  $B_h$  and again our statement is true. The minimum property is proved similarly.



Exercises: 1. Carry out these two procedures for the square of figure 6. Compare the results with those obtained by the method described on page 85. Also compare the value of  $u$  for the center of the square with that found in exercise c of section 4, Part II of this chapter.

2. Repeat the numerical calculations with a finer mesh of 49 mesh points, as in figure 10, and compare the results with those of exercise 1.

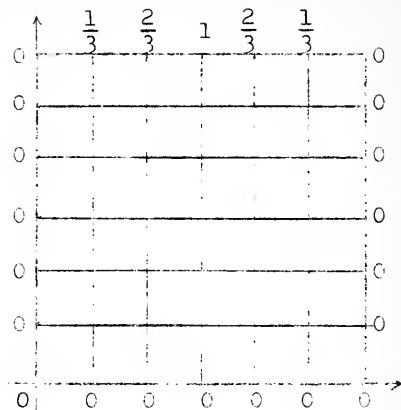


Fig. 10

### 3. Theoretical Remarks

a) Uniqueness of the solution: From the mean value property of the solution of the difference equation we saw that the maximum and minimum values of  $u$  certainly are attained on the boundary  $C_h$  of  $B_h$ . (See footnote on page 87). It follows that the boundary value problem

$$(9) \quad \begin{cases} \nabla_h^2 u = 0 & \text{in } B_h \\ u = 0 & \text{on } C_h \end{cases}$$

has the solution

$$u = 0 \quad \text{in } B_h.$$

Suppose, then, that we have a boundary value problem

$$(10) \quad \begin{cases} \nabla_h^2 u = 0 & \text{in } B_h \\ u = g(P) & \text{on } C_h \end{cases}$$

for which there exist two solutions. The difference of these two solutions is then a solution of the boundary value problem (9), namely, their difference is identically zero, whence they are equal. Thus, the solution to the general problem (10) is unique.





b) Convergence of the methods of section 2: The proofs of the convergence of the two procedures described in section 2 are omitted here. It should be pointed out, however, that these proofs follow very easily from the consideration of a minimum property to be described in paragraph d) of this section\*.

c) Convergence of the difference equation to a differential equation: We merely remark that if we let the mesh  $h$  of  $B_h$  tend to zero, then, naturally,  $B_h \rightarrow B$  and  $C_h \rightarrow C$ , and, furthermore, the difference equation  $\nabla_h^2 u = 0$  tends to the differential equation  $\nabla^2 u = 0$  in  $B$ . Likewise, the boundary values along  $C_h$  approach the boundary values along  $C$  and the solution of the boundary value problem of the difference equation tends to the solution of the corresponding boundary value problem for the differential equation\*\*. Analogous statements are true also in the case of plates\*\*.

d) Connection with a minimum problem: The key to a more detailed study of the Laplace difference equation is given by its connection with certain minimum or variational problems. We first recall the well-known fact, from the principle of least squares, that the value  $u$  for which the sum of the squares of the deviations of  $m$  values  $u_1, \dots, u_m$  from  $u$ ,

$$\sum_{j=1}^m (u - u_j)^2$$

is a minimum, is given by the arithmetic mean

$$u = \frac{u_1 + u_2 + \dots + u_m}{m}.$$

Applying this remark to our difference equation  $\nabla_h^2 u = 0$  we discover immediately that the solution to our boundary value problem is a function  $u(x, y)$ , defined in the given net domain and satisfying the prescribed boundary values, for which the sum

---

\* For the actual proofs see R. Courant: "Über Randwertaufgaben bei partiellen Differenzgleichungen" which appears in the "Zeitschrift für angewandte Mathematik und Mechanik, No. 6, (1925). See especially pages 322-325.

\*\* See R. Courant, K. Friedrichs, and H. Levy: "Über partiellen Differenzgleichungen der Mathematischen Physik" appearing in the "Mathematische Annalen", Vol. 100, (1928).



$$(11) \quad \frac{1}{2} \sum \sum [u(P) - u(Q)]^2$$

is as small as possible, the summation being extended over all pairs of neighboring net points in  $B_h$  including the boundary points.

This fact indicates that the "harmonic" net function is the "smoothest" of all possible net functions defined in  $B_h$  and having the same boundary values. Its smoothness at any point is measured by the squares of the deviations of the neighboring values of the function.

To establish the connection with classical results we remark that, as the dimension  $h$  of  $B_h$  tends to zero, difference quotients tend to derivatives and summations become integrals. We may write (11) in the form

$$\frac{1}{2} h^2 \sum \sum \frac{[u(P) - u(Q)]^2}{h^2}$$

or, using our previous notations,

$$\frac{1}{2} h^2 \sum \sum (u_x^2 + u_x^2 + u_y^2 + u_y^2)^* .$$

As  $h \rightarrow 0$ , this expression tends to the integral

$$(12) \quad \iint_B \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dx dy .$$

Corresponding to our statement concerning the sum (11) we have, here, the classical Gauss-Thompson-Dirichlet principle, which states: "The harmonic function satisfying the differential equation  $\nabla^2 u = 0$  in  $B$  is that function, of all possible 'admissible' functions assuming the same boundary values, for which the integral (12) is a minimum". We have not stated what are to be considered admissible functions; we may, for example, admit all functions which are continuous and which possess continuous first and second derivatives.

---

\* The terms of the expression  $[u(P) - u(Q)]^2$  are actually counted twice, once forward and once backward.



We have not actually proved the above principle, but, rather, have attempted to make it seem plausible. We shall see later that the formulation of various problems as variational problems leads to a very powerful method for performing numerical calculations - a method which is widely applied in dealing with engineering problems - the so-called Rayleigh-Ritz method.

In the present section our discussion was confined to the homogeneous difference equation. It applies, however, just as well to the non-homogeneous equation

$$(13) \quad \nabla_h^2 u = f(x, y)$$

where  $f(x, y)$  is a given net function in  $B_h$ .

Exercise: By one of the methods employed for the homogeneous equation solve the boundary value problem

$$\begin{cases} \nabla_h^2 u = 1 & \text{in a square} \\ u = 0 & \text{on the sides of the square} \end{cases}$$

Use a mesh with 49 net points. Find, in particular, the value of  $u$  at the center of the square.

#### Part IV. Non-homogeneous Equilibrium Equation. Green's Function

##### 1. The Non-homogeneous and Homogeneous Equations

The equilibrium problem for the elastic membrane as formulated in section 2, Part I, of this chapter, refers, in general, to the non-homogeneous partial differential equation

$$(1) \quad \nabla^2 u = f(x, y)$$

defined in a domain  $B$ , together with the prescribed non-homogeneous boundary function

$$(2) \quad u = g(s) \quad \text{on } C$$



where  $s$  is the arc length measured along the boundary curve  $C$ . We wish to show now that this general problem can always be replaced by a less general problem, i.e., one in which the non-homogeneity enters either in the differential equation or in the boundary condition, but not in both.

Let the boundary curve  $C$  be given by the equations  $x = x(s)$ ,  $y = y(s)$ , where the parameter  $s$  is the arc length on  $C$ . Then consider a known function  $g(x,y)$  defined throughout  $B$  and  $C$ , which is such that  $g(x(s),y(s)) = \dot{g}(s)$  where  $g(s)$  is the boundary function prescribed on  $C$ . That is,  $g(x,y)$  is to coincide with  $g(s)$  on  $C$ . Also, suppose  $g(x,y)$  to possess continuous first and second derivatives. Returning to equations (1) and (2), let  $u(x,y) = w(x,y) + g(x,y)$  where  $w$  is an unknown function. Substituting this expression for  $u$  into (1) and (2), we get a new boundary value problem, this time for the function  $w$ . The boundary condition is

$$(3) \quad w = 0 \quad \text{on } C$$

since  $u = g(x,y)$  on  $C$ , and the differential equation is

$$(4) \quad \nabla^2 w = f - \nabla^2 g = F(x,y)$$

where  $F(x,y)$  is, of course, a known function, since  $f$  and  $g$  and, hence,  $\nabla^2 g$  are known. Thus the general problem given by (1) and (2) is reduced to a simpler one given by (3) and (4) where the non-homogeneity enters only in the differential equation and not in the boundary condition. Incidentally, the non-homogeneous differential equation is, as we have mentioned before, the so-called Poisson equation.

On the other hand, it can be shown that it is always possible to reduce the non-homogeneous Poisson equation

$$\nabla^2 u = f(x,y)$$

to the homogeneous Laplace equation. To this end let  $p(x,y)$  be any known particular solution of the Poisson equation. Writing  $u(x,y) = v(x,y) + p(x,y)$ , where  $v$  is an unknown function, we have





$$\nabla^2 u = \nabla^2 v + \nabla^2 p$$

or

$$f = \nabla^2 v + f$$

whence

$$(5) \quad \nabla^2 v = 0.$$

Since  $u = g(s)$  on  $C$ , it follows from  $v = u - p$  that

$$(6) \quad v = g(s) = p(C) \quad \text{on } C$$

where  $p(C)$  is the known value of  $p(x,y)$  along  $C$ . Thus the general problem given by (1) and (2) has been reduced to one, given by (5) and (6), in which the non-homogeneity occurs solely in the boundary condition and not in the differential equation.

In the next section we shall show a method for finding particular solutions  $p(x,y)$  of the Poisson equation. We may mention, however, at this point, a well-known theorem of potential theory concerning a particular solution of this differential equation. The theorem states that the function

$$(7) \quad u(x,y) = -\frac{1}{2\pi} \int_B f(\xi, \eta) \log r \, d\xi \, d\eta$$

satisfies the Poisson equation

$$(1) \quad \nabla^2 u = f(x,y)$$

where  $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$  is the distance between any two points  $P:(x,y)$  and  $Q:(\xi, \eta)$  of  $B$ , the integration being performed over all points  $Q:(\xi, \eta)$  of  $B$ .

Furthermore, it is easily seen that the function

$$q(x,y; \xi, \eta) = \log r$$

satisfies the Laplace equation

$$\nabla^2 q = 0 \quad \text{at all points of } B$$

except where  $P = Q$ .\*

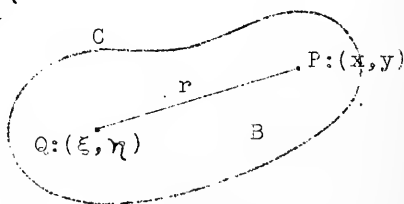


Fig. 1

\* The operation  $\nabla^2$  may be carried out with respect to  $x$  and  $y$  or with respect to  $\xi$  and  $\eta$ , since  $q(x,y; \xi, \eta)$  is symmetric, i.e.,

$$q(x,y; \xi, \eta) = q(\xi, \eta; x,y).$$



## 2. The Boundary Value Problem for the Poisson Equation in a Square Domain

We consider in this section the boundary value problem given by

$$(1) \quad \nabla^2 u = f(x, y) \quad \text{in } B$$

$$(8) \quad u = 0 \quad \text{on } C$$

where the domain  $B$  is the square

$$(9) \quad \begin{cases} 0 \leq x \leq \pi a \\ 0 \leq y \leq \pi a \end{cases}.$$

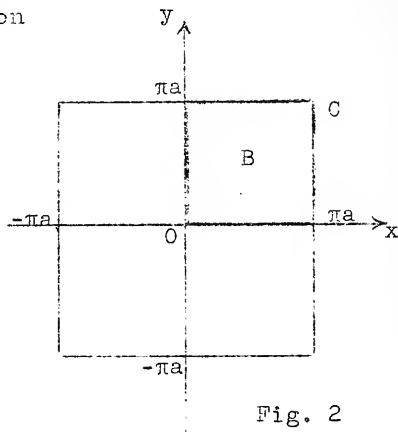


Fig. 2

Let the function  $f(x, y)$ , originally defined in  $B$ , be continued over the square

$$\begin{cases} -\pi a \leq x \leq \pi a \\ -\pi a \leq y \leq \pi a \end{cases}$$

as an odd function in both  $x$  and  $y$ . Assume, too, that  $f(x, y)$  is sufficiently smooth to admit of a Fourier expansion which converges uniformly everywhere except, possibly, on the boundary of  $B$ . Then, since  $f(x, y)$  is an odd function in the large square, the Fourier series will be in a pure sine series, i.e.,

$$(10) \quad f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{nm} \sin \frac{nx}{a} \sin \frac{my}{a} \quad *$$

---

\* We may regard a Fourier series for a function of two variables as obtained from that for a function of one variable as follows. First, expand  $f(x, y)$  in terms of  $x$ . The Fourier coefficients are then functions of  $y$ . Expand these coefficients as Fourier series in  $y$ . Then, combining terms, we arrive at the Fourier series for  $f(x, y)$  in terms of  $x$  and  $y$  together. The series, in this case, is actually a double series, since there appears a summation over  $n$  and also over  $m$ . For a more detailed discussion see Courant-Hilbert: "Methoden der Mathematischen Physik", Vol. I, page 62.



where

$$(11) \quad \alpha_{nm} = \frac{4}{\pi^2 a^2} \int_0^{\pi a} \int_0^{\pi a} f(\xi, \eta) \sin \frac{n\xi}{a} \sin \frac{m\eta}{a} d\xi d\eta.$$

Equation (10) immediately suggests the form of the solution to the boundary value problem, i.e.,

$$(12) \quad u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{nx}{a} \sin \frac{my}{a},$$

where the  $A_{nm}$  are constants to be determined. An inspection of (12) reveals that it satisfies the boundary condition (8) for any values of these constants. Hence, there remains only to determine the  $A_{nm}$  so that (12) will satisfy the differential equation (1). We apply the Laplace operator to (12), performing the differentiation term by term. (We assume this to be permissible). We obtain

$$(13) \quad \nabla^2 u = -\frac{1}{a^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} (n^2 + m^2) \sin \frac{nx}{a} \sin \frac{my}{a}.$$

Immediately, then, we have, by comparing with (10),

$$(14) \quad A_{nm} = -\frac{a^2 \alpha_{nm}}{n^2 + m^2}.$$

Substituting this value for  $A_{nm}$  into (12), we obtain the desired solution to our boundary value problem,

$$(15) \quad u(x, y) = -a^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} \frac{\sin \frac{nx}{a} \sin \frac{my}{a}}{n^2 + m^2}.$$

It is possible to write this solution in a different form, one that will lead us to a general method for solving boundary value problems, not only for square domains but also for more general ones. Substitute in (15) the values of the  $\alpha_{nm}$  given by (11), thus

$$u(x, y) = -\frac{4}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{nx}{a} \sin \frac{my}{a}}{n^2 + m^2} \int_0^{\pi a} \int_0^{\pi a} f(\xi, \eta) \sin \frac{n\xi}{a} \sin \frac{m\eta}{a} d\xi d\eta.$$



Interchanging summation and integration (without attempting to justify such a step), we obtain

$$(16) \quad u(x,y) = -\frac{4}{\pi^2} \int_0^{\pi a} \int_0^{\pi a} f(\xi, \eta) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{n x}{a} \sin \frac{m y}{a} \sin \frac{n}{a} \sin \frac{m}{a}}{n^2 + m^2} d\xi d\eta.$$

We observe that the double sum in the integrand is a function of two pairs of variables,  $x, y$  and  $\xi, \eta$ . By writing

$$(17) \quad K(x,y; \xi, \eta) = -\frac{4}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin \frac{n x}{a} \sin \frac{m y}{a} \sin \frac{n \xi}{a} \sin \frac{m \eta}{a}}{n^2 + m^2},$$

equation (16) becomes

$$(18) \quad u(x,y) = \iint_B K(x,y; \xi, \eta) f(\xi, \eta) d\xi d\eta,$$

where

$$\iint_B \dots d\xi d\eta \int_0^{\pi a} \int_0^{\pi a} \dots d\xi d\eta.$$

A few general remarks may be made. First of all, it is clear that  $K$  is symmetric in the two pairs of variables, i.e.,

$$K(x,y; \xi, \eta) = K(\xi, \eta; x,y).$$

Secondly, although we have omitted the question of convergence of the double series, we might mention an important fact concerning it, namely, if  $\xi = x$  and  $\eta = y$ , then the value of the series becomes infinite. In other words, if  $P$  and  $Q$  are any two points in  $B$ , with coordinates  $(x,y)$  and  $(\xi, \eta)$ , respectively, then  $K(P;Q) \rightarrow \infty$  as  $P \rightarrow Q$ . If, on the other hand,  $P \neq Q$ , then the convergence of the series is assured when we consider the limit  $N \rightarrow \infty$  of the finite partial sums,  $n$  and  $m$  ranging from 1 to  $N$ .

Finally, we notice that  $K(x,y; \xi, \eta)$  does not depend in any way upon the function  $f(x,y)$ . This is a significant point which we shall follow up in the next section.  $K(x,y; \xi, \eta)$  is called the Green's function of the differential equation  $\nabla^2 u = 0$  for the square (9).





### 3. Equilibrium under Concentrated External Forces

We consider here, in a more or less abstract fashion, a class of problems  $P$  of which the problem discussed in the previous section is a typical example. The class  $P$  is defined by the Laplace operator  $\nabla^2$ , a given domain  $B$ , and a given boundary function, say  $u = 0$ , on the boundary curve  $C$ . The individual members of the class are characterized by the function  $f$  which appears in the equation  $\nabla^2 u = f$ . Any given function  $f$  gives rise to a particular problem of the class  $P$ . We may employ the convenient functional notation  $P = P(f)$  indicating that for every choice of the function  $f$  there corresponds a particular problem of the class  $P$ . The general statement of a problem of the class  $P$  is given by

$$(19) \quad \begin{cases} \nabla^2 u = f & \text{in } B \\ u = 0 & \text{on } C \end{cases} .$$

The solution to this problem is given by a general formula which involves a function  $K(x, \dots; \xi, \dots)$  of the variables  $x, \dots$  occurring in (19) and of the same number of space variables  $\xi, \dots$  whose significance will soon become evident. The function  $K$  depends only on the class  $P$ , i.e., on the differential operator considered, the domain  $B$ , and the boundary function. It does not depend upon the function  $f$ .  $K$  thus represents or characterizes the class  $P$ , and the form of the solution illustrates in what number  $P$  is represented by  $K$  and also makes evident the role played by the function  $f$ .

A situation of this kind appears not only in this and in other types of boundary value problems, but also in many other types of problems of both pure and applied mathematics.

We now proceed to make our general and abstract statements more precise and concrete and at the same time to reveal the intimate connection with a very natural concept of physics, that of concentrated external force.

Consider a physical system capable of motion about a state of stable equilibrium. For example, the system may be an elastic string or cantilever characterized by one space

2. Hamiltonian under Canonical Transformations

Let  $H$  be a Hamiltonian function,  $q$  and  $p$  the generalized coordinates and momenta,  $t$  the time. The Hamiltonian function is a function of  $q, p$  and  $t$ . The equations of motion are given by the Hamiltonian equations:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

where  $\dot{q}$  and  $\dot{p}$  are the time derivatives of  $q$  and  $p$  respectively. The Hamiltonian function is a constant of motion, i.e.,  $H$  is constant along the trajectory of the system.

Consider a canonical transformation  $(q, p) \rightarrow (Q, P)$ . The new Hamiltonian  $K$  is given by:

$$K(Q, P, t) = H(q, p, t) + \frac{dF}{dt}$$

where  $F$  is the generating function of the transformation. The equations of motion in the new coordinates are:

$$\dot{Q} = \frac{\partial K}{\partial P}, \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

(10)

The Hamiltonian function is a function of the generalized coordinates and momenta. The equations of motion are given by the Hamiltonian equations. The Hamiltonian function is a constant of motion. Consider a canonical transformation. The new Hamiltonian is given by the old Hamiltonian plus the time derivative of the generating function. The equations of motion in the new coordinates are given by the Hamiltonian equations with the new Hamiltonian.

The Hamiltonian function is a function of the generalized coordinates and momenta. The equations of motion are given by the Hamiltonian equations. The Hamiltonian function is a constant of motion. Consider a canonical transformation. The new Hamiltonian is given by the old Hamiltonian plus the time derivative of the generating function. The equations of motion in the new coordinates are given by the Hamiltonian equations with the new Hamiltonian.

The Hamiltonian function is a function of the generalized coordinates and momenta. The equations of motion are given by the Hamiltonian equations. The Hamiltonian function is a constant of motion. Consider a canonical transformation. The new Hamiltonian is given by the old Hamiltonian plus the time derivative of the generating function. The equations of motion in the new coordinates are given by the Hamiltonian equations with the new Hamiltonian.

coordinate  $x$ ; or it may be an elastic membrane or plate\* characterized by two space variables  $x, y$ . Again, the system might be a three-dimensional body with coordinates  $x, y, z$ . For purposes of brevity and generality we denote the coordinates of a point by  $(x, \dots)$ . Thus, for a string  $(x, \dots) = (x)$ , and for a membrane  $(x, \dots) = (x, y)$ . With this notation, we let  $u(x, \dots)$  be the deflection of the point  $(x, \dots)$  from its equilibrium position. Since only static states of physical systems are to be considered, the deflection  $u$ , which is the result of an external force, depends only on the point  $(x, \dots)$  and not on the time  $t$ . Furthermore, since we are dealing with continuous bodies, we have to consider the density of masses and forces. Thus, for example, in (19),  $f(x, y)$  represents the density of the external force. The total exterior force  $F$  acting on a system is given by integrating  $f$  over the domain  $B$ , thus

$$F = \int_B \dots \int f(\xi, \dots) d\xi \dots \dots$$

In mechanics, the idea of a mass concentrated at a single point is a convenient idealization. Likewise, it is convenient to consider the external forces as being concentrated at single points of a physical system. Assume, for simplicity, that the intensity of a concentrated force at a point  $Q: (\xi, \dots)$  is unity. The force at  $Q$  will cause a deflection at each point  $P: (x, \dots)$  of the system. This deflection will depend upon the point  $P$  under observation and also upon the point  $Q$  at which the unit force is applied. We denote the deflection appropriately by

$$(20) \quad K(x, \dots; \xi, \dots)$$

The function  $K$  is called Green's function of the system in question.

Green's function does not depend upon density  $f$  of the external force. Rather, it satisfies the homogeneous differential equation  $\nabla^2 u = 0$  at all points  $(x, \dots)$  with the

---

\* We mention a plate although the motion or equilibrium of this system is not characterized by the operator  $\nabla^2$  but by  $\nabla^2 \nabla^2$ . The general considerations, however, are the same.



exception of the point of application ( $\xi, \dots$ ) of the concentrated force. At this point the right-hand side of the differential equation, i.e., the density of the external force, so to speak, becomes infinite, since the total force at this point is assumed to be unity. At such a point we should expect  $K$  to possess a singularity. Hence, we may say that Green's function  $K(P;Q)$  (where  $P = (x, \dots)$  and  $Q = (\xi, \dots)$ ) is a solution of the homogeneous boundary value problem (i.e., problem (19) with  $f = 0$ ) for all  $P \neq Q$ . For  $P = Q$  the Green's function has a singularity which we shall presently characterize.

We make still another remark, the truth of which will become evident in the particular cases to be discussed. This is the observation we made in the case of the square, that the Green's function is symmetric,\*

$$(21) \quad K(x, \dots; \xi, \dots) = K(\xi, \dots; x, \dots)$$

or

$$(21') \quad K(P;Q) = K(Q;P) \quad .$$

Physically, this expresses a law of reciprocity which occurs frequently in physics: If an external force of unit intensity applied at a point  $Q$  of a system results in an effect of amount  $K(P;Q)$  at a point  $P$ , then a force of unit intensity applied at  $P$  has the same effect  $K(P;Q)$  at the point  $Q$ .

The general form of the solution to the non-homogeneous boundary value problem of class  $P$  with a continuously distributed external force density  $f(x, \dots)$  is given by the integral

$$(22) \quad u(x, \dots) = \int_B \dots \int K(x, \dots; \xi, \dots) f(\xi, \dots) d\xi \dots \quad .$$

In other words, the resulting deflection  $u$  is the superposition of the effects of single concentrated forces acting at each point  $(\xi, \dots)$  of the system in the domain  $B$ . The individual concentrated forces do not have unit intensity but rather the intensity  $f(\xi, \dots) \Delta \xi \dots$  .

---

\* This is not so, however, in the event that a dissipative force is present.



This is the general pattern of the concept of Green's function. For our purposes, there is no point in giving precise mathematical proofs or in investigating such theoretical questions as the "existence" of Green's function. But we do wish to understand its fundamental and intuitive significance and its use in applications. Hence, we shall discuss a few special cases.

#### 4. Green's Function for the Elastic String

We have seen in an earlier section that the boundary value problem for the equilibrium of an elastic string of length  $\ell$  with ends fixed on the  $x$ -axis, is given by

$$(23) \quad \begin{cases} u'' = f(x) & (0 \leq x \leq \ell) \\ u(0) = u(\ell) = 0 \end{cases} .$$

To obtain Green's function for the string we suppose a concentrated unit force to be applied at a point  $x = \xi$ . A deflection will occur, the shape of which is shown in figure 3 to be two straight lines meeting at a vertex which lies on the

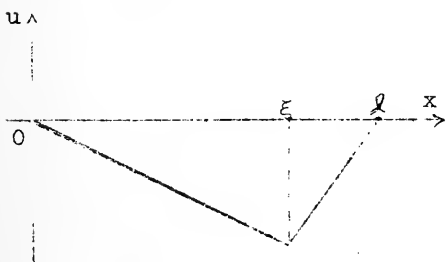


Fig. 3

line  $x = \ell$ . That these two segments are straight is seen in the fact that Green's function satisfies  $u'' = 0$  for all  $x \neq \xi$ . Hence  $u$  is a linear function.

Green's function is completely found as soon as we determine the location of the vertex formed by the two straight segments. To do this it is necessary to investigate more exactly the singularity of

Green's function at  $x = \xi$ . We suppose the external force not to be concentrated at  $x = \xi$  but to be distributed over a small interval  $(\xi - \varepsilon \leq x \leq \xi + \varepsilon)$  of width  $2\varepsilon$  about  $x = \xi$ . Letting  $\varepsilon \rightarrow 0$  yields, in the limit, the concentrated force considered





above. If we denote the density of the distributed force by  $\varphi_\varepsilon(x)$ , then  $\varphi_\varepsilon(x)$  is defined as follows. For points outside the interval  $(\xi - \varepsilon \leq x \leq \xi + \varepsilon)$ ,  $\varphi_\varepsilon(x) = 0$ , while inside the interval  $\varphi_\varepsilon(x)$  is such that

$$\lim_{\varepsilon \rightarrow 0} \int_{\xi - \varepsilon}^{\xi + \varepsilon} \varphi_\varepsilon(x) dx = 1.$$

For example, we may take  $\varphi_\varepsilon(x) = 1/2\varepsilon$  inside the interval, as in figure 4.

The equilibrium equation is now

$$(24) \quad u'' = \varphi_\varepsilon(x).$$

Integrating both sides of this equation from  $x = \xi - \varepsilon$  to  $x = \xi + \varepsilon$  and letting  $\varepsilon \rightarrow 0$ , we get

$$(25) \quad \lim_{\varepsilon \rightarrow 0} [u'(\xi + \varepsilon) - u'(\xi - \varepsilon)] = 1.$$

This actually completes the characterization of Green's function; for it states that the derivative at  $x = \xi$  must have a jump discontinuity of the amount 1. In other words, the two straight segments of figure 3 meet on the line  $x = \ell$  at such a point that the difference of the slopes of the two lines is exactly 1. This point is thus uniquely determined.

Explicitly, Green's function for the string is

$$(26) \quad K(x; \xi) = \begin{cases} (\xi - \ell) \frac{x}{\ell} & \text{for } x \leq \xi \\ (x - \ell) \frac{\xi}{\ell} & \text{for } x \geq \xi \end{cases}.$$

It is characterized mathematically by the following statements:

a)  $K(x; \xi)$  is a continuous function of  $x$  for a fixed  $\xi$  and satisfies the boundary conditions of (23), namely,

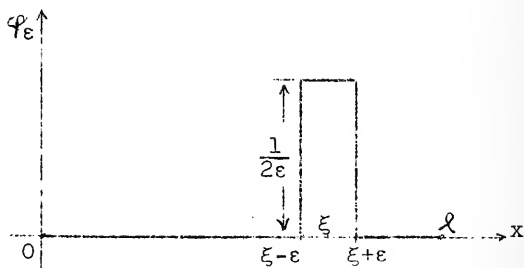


Fig. 4



$$K(0; \xi) = K(\ell; \xi) = 0.$$

b)  $K(x; \xi)$  has a single jump discontinuity in the first derivative,

$$\lim_{\varepsilon \rightarrow 0} \left. \frac{dK}{dx} \right|_{x=\xi-\varepsilon}^{x=\xi+\varepsilon} = 1.$$

c)  $K(x; \xi)$  satisfies the homogeneous differential equation  $u'' = 0$  everywhere except at  $x = \xi$ .

We notice, also, that  $K$  is symmetric, i.e.,  $K(x; \xi) = K(\xi; x)$ .

Finally, according to the general pattern of the preceding section, the solution of the boundary value problem (23) is

$$(27) \quad u(x) = \int_0^\ell K(x; \xi) f(\xi) d\xi,$$

where  $K(x; \xi)$  is given by (26).

Exercise: Verify the last statement directly.

## 5. Green's Function for the Elastic Membrane

Consider an elastic membrane stretched over a bounded domain  $B$  of the  $x, y$ -plane. Let  $C$  be the piece-wise smooth boundary curve of  $B$  as usual. If we suppose the membrane to be fixed along  $C$ , then the equilibrium boundary value problem is

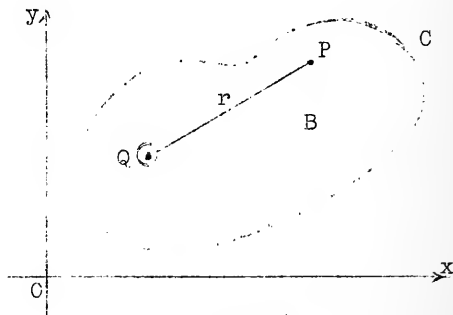


Fig. 5

$$(28) \quad \begin{cases} \nabla^2 u = f(x, y) & \text{in } B \\ u = 0 & \text{on } C \end{cases}.$$



We attempt to find Green's function for this membrane. We assume a single concentrated unit force to be acting at a point  $Q(\xi, \eta)$  of the membrane and no other external force elsewhere. We accomplish this concentrated force by distributing an external force over a small circle  $K_\varepsilon$  of radius  $\varepsilon$  about  $Q$  and then allowing  $\varepsilon$  to tend to zero. Let the density of the distributed force be denoted by  $\varphi_\varepsilon(x, y)$ . Then  $\varphi_\varepsilon = 0$  in  $B$  except in the interior of  $K_\varepsilon$  where it should be such that

$$\lim_{\varepsilon \rightarrow 0} \int \int_{K_\varepsilon} \varphi_\varepsilon(x, y) \, dx \, dy = 1 \quad .^*$$

Thus, for example, we may take  $\varphi_\varepsilon = 1/\pi\varepsilon^2$  in  $K_\varepsilon$ .

The equilibrium equation for the case where the force density is  $\varphi_\varepsilon(x, y)$  is

$$(29) \quad \nabla^2 u = \varphi_\varepsilon(x, y) \quad .$$

If  $\varepsilon \rightarrow 0$  we obtain the

desired concentrated force

at  $Q$  and the solution  $u = K(x, y; \xi, \eta)$  of the limiting equation of (29) with the boundary condition of (28) is the desired Green's function.

To obtain  $K(x, y; \xi, \eta)$  we integrate both sides of (29) over the interior of any circle  $\Gamma$  in  $B$  of radius  $r \geq \varepsilon$ , thus

$$\int \int_{\Gamma} \nabla^2 u \, dx \, dy = \int \int_{K_\varepsilon} \varphi_\varepsilon \, dx \, dy \quad .$$

Applying Green's formula to the left side of this equation, we get

$$\int_{\Gamma} \frac{\partial u}{\partial r} \, ds = \int \int_{K_\varepsilon} \varphi_\varepsilon \, dx \, dy$$

---

\* The notation  $\int \int_{K_\varepsilon}$  means "the interior of  $K_\varepsilon$ ".

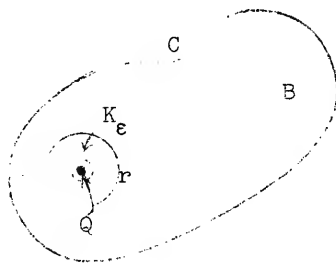


Fig. 6



where  $\partial/\partial r$  means differentiation in the direction of the radius  $r$  of  $\Gamma$  and  $s$  is the arc length along  $\Gamma$ . As  $\epsilon \rightarrow 0$  the right side of the last equation tends to 1 and, hence, of Green's function it must be true that

$$(30) \quad \int_{\Gamma} \frac{\partial K}{\partial r} ds = 1.$$

Equation (30) is satisfied by  $\frac{1}{2\pi} \log r$ . Hence, we may characterize Green's function for a membrane by

$$(31) \quad K(x, y; \xi, \eta) = \frac{1}{2\pi} \log r + \gamma(x, y; \xi, \eta),$$

where  $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$  and  $\gamma(x, y; \xi, \eta)$  is a regular function satisfying  $\nabla^2 \gamma = 0$ . Also,  $K = 0$  on the boundary  $C$ .

We see that in the case of a two-dimensional domain Green's function for the operator  $\nabla^2$  is not explicitly expressible as it was in the case of the string. The reason for this is that in one dimension there is essentially only one type of domain, the interval, whereas in two dimensions there is a large variety of shapes of domains, e.g., the square, circle, ellipse, etc. The function  $\gamma(x, y; \xi, \eta)$  depends upon the shape of the domain over which the membrane is stretched.

In the case of the string the singularity appeared only in the first derivative. The membrane suffers a much stronger singularity under a concentrated force, namely; the deflection becomes infinite of the order of the logarithm of the distance between the point of application  $Q$  of the force and the point of observation  $P$  as  $P \rightarrow Q$ .

The fact that the deflection actually becomes infinite may seem surprising in comparison with the case of the string where the deflection under a single concentrated force suffered no discontinuity. The explanation of this phenomenon may be indicated briefly by means of energy considerations. In the mathematical string\* the potential energy is proportional to the

---

\* It must be remembered that the string and the membrane were defined mathematically here, that is, they are "objects" that satisfy certain differential equations. They are idealizations of the actual physical string and membrane.





change in length of the string under deflection. We know that the deflection under a single concentrated external force takes the form of two straight lines issuing from the end-points. Thus, if the deflection became infinite at  $x = \xi$ , the length of the string, and hence the potential energy, would become infinite. On the other hand, in the case of the mathematical membrane, an infinite deflection may very well occur, as one knows, without changing the area of the membrane by any great amount. Therefore, the potential energy, which is proportional to the change of area, would remain finite.

In conclusion we give the complete characterization of Green's function  $K(x, y; \xi, \eta)$  for the elastic membrane in a two-dimensional domain B.

- a)  $K$  and its first and second derivatives are continuous in B except for  $r = 0$ . It has the form given by (31).
- b)  $K = 0$  on the boundary curve C.
- c)  $K$  satisfies the equation  $\nabla^2 K = 0$  except for  $r = 0$ .

Again it may be seen that Green's function is symmetric,  $K(x, y; \xi, \eta) = K(\xi, \eta; x, y)$ . Finally, according to our general pattern, the solution of the boundary value problem given by (28) is

$$(32) \quad u(x, y) = \iint_B K(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

Exercise: Verify that (32) solves the boundary value problem (28).\*

## 6. Further Examples of Green's Function

Having discussed Green's function for the elastic string and membrane more or less at length, we wish to go still further. We can do so in three different ways. Firstly, we may consider problems in domains of higher dimension, e.g., an elastic body in three dimensions. Secondly, we could investigate other differential operators than the Laplace operator. Such would be

---

\* See Courant-Hilbert: Methoden der Mathematischen Physik., Vol. I, pages 316 ff.



the case in problems dealing with cantilevers or plates. Thirdly, we may consider less simple types of boundary conditions than  $u = 0$ .

As an example of the extension to more dimensions, consider a three-dimensional elastic body in a domain  $B$  bounded by a regular surface  $C$ . The equilibrium equation for the deflection  $u(x,y,z)$  under an external force of density  $f(x,y,z)$  is

$$(33) \quad \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = f(x,y,z) .$$

If we impose the boundary condition

$$(34) \quad u = 0 \quad \text{on the surface } C,$$

then Green's function is

$$(35) \quad K(x,y,z; \xi, \eta, \zeta) = \frac{1}{4\pi r} + \delta(x,y,z; \xi, \eta, \zeta) ,$$

where  $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ , the distance between a point  $P:(x,y,z)$  and the point  $Q:(\xi, \eta, \zeta)$  of application of the concentrated force. The function  $\delta(x,y,z; \xi, \eta, \zeta)$  is a regular function satisfying  $\nabla^2 \delta = 0$  in  $B$ . We notice that the singularity of  $K$  is greater for the three-dimensional membrane than for the two-dimensional one.

A cantilever is a one-dimensional body which resists bending but offers no resistance to stretching. Its equilibrium equation under an impressed force of density  $f(x)$  involves a fourth order differential operator, thus

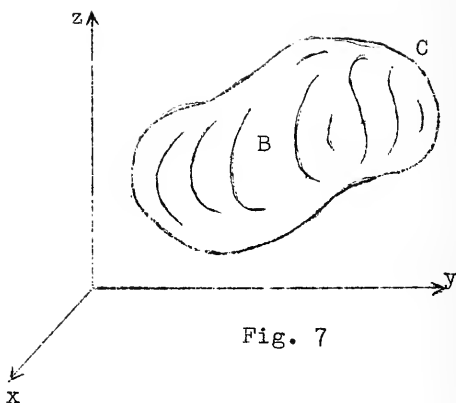


Fig. 7

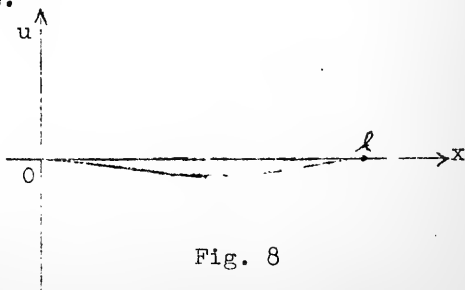


Fig. 8



$$(36) \quad \nabla^2 \nabla^2 u = \frac{d^4 u}{dx^4} = u'''' = f(x) \quad .$$

If we impose the boundary conditions

$$(37) \quad \begin{cases} u(0) = u(\ell) = 0 \\ u'(0) = u'(\ell) = 0 \end{cases} \quad ,$$

by clamping the ends of the cantilever, then Green's function  $K(x; \xi)$  for the cantilever has a characteristic singularity in the third derivative, i.e.,

$$(38) \quad \lim_{\varepsilon \rightarrow 0} \frac{d^3 K(x; \xi)}{dx^3} \bigg|_{x=\xi-\varepsilon}^{x=\xi+\varepsilon} = 1 \quad .$$

The singularity here is less marked than that which occurs in the case of a string since, for the cantilever,  $K$  and its first and second derivatives are continuous in the interval  $0 \leq x \leq \ell$ .

A plate is a two-dimensional body which, like the cantilever, resists bending but offers no resistance to stretching. For it, also, the equilibrium equation is of the fourth order,

$$(39) \quad \nabla^2 \nabla^2 u = f(x, y) \quad .$$

Imposing the boundary conditions

$$(40) \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } C \quad ,$$

where  $n$  is the normal direction to the boundary curve  $C$ , we find the Green's function to be less singular than that for the membrane. It is

$$(41) \quad K(x, y; \xi, \eta) = \frac{1}{2\pi} r^2 \log r + \gamma(x, y; \xi, \eta) \quad .$$

As a matter of fact, the singularity in  $K$  does not appear until we consider the second derivative with respect to  $r$ . Here the singularity is logarithmic. The function  $\gamma(x, y; \xi, \eta)$ , of course, satisfies the equation  $\nabla^2 \nabla^2 \gamma = 0$ .



As an example of a problem with a non-simple boundary condition consider an elastic string with one end fixed at  $x = 0$  and the other free to move along the line  $x = \ell$ . The boundary conditions are

$$(42) \quad u(0) = 0, \quad u'(\ell) = 0.$$

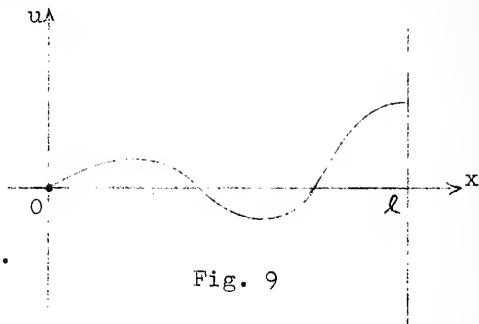


Fig. 9

Green's function in this case may be given explicitly.

It is

$$(43) \quad K(x; \xi) = \begin{cases} x & \text{for } x \leq \xi \\ \xi & \text{for } x \geq \xi \end{cases}.$$

Exercises: 1) Find expression (35) for Green's function for a three-dimensional elastic body by considering a single concentrated external force of unit intensity acting at a point  $(\xi, \eta)$ .

2) Do the same for the plate and cantilever. For the latter calculate  $K$  explicitly.

3) Carry out the calculations in the case of the string discussed above and obtain (43).

4) In the three examples above, prove that for an external force density  $f(x, \dots)$ ,

$$u(x, \dots) = \int_B \int K(x, \dots; \xi, \dots) f(\xi, \dots) d\xi \dots$$

satisfies the corresponding boundary value problem.





## 7. Realization of Green's Function by Physical Means

Although Green's function is a very elegant mathematical concept, it is not quite so simple a matter to calculate it in a practical way. Even in the case of the elementary membrane,  $K$  appears as a function of four variables. However, Green's function was presented to us as a natural physical phenomenon. Because of this it can be interpreted and, in some cases, actually found in a manner which lends itself to numerical or graphical evaluation by experimental means.

Consider, for example, a three-dimensional closed chamber  $B$  whose inside wall  $C$  is covered with an electrical conductor, say tin foil. If we apply a unit electric charge at a point  $Q$  inside  $B$ , an electric potential will establish itself throughout  $B$ . This electric potential will be the Green's function for the elastic body problem with operator  $\nabla^2$ , domain  $B$ , and boundary condition  $u = 0$  on  $C$ . For, as is shown in electrostatic theory, the boundary must be at a state of equilibrium and, therefore, at constant potential, say zero, while the single charge at  $Q$  yields a singularity of magnitude  $1/4\pi r$ , where  $r$  is the distance between  $Q$  and another point  $P$  of  $B$ .

Green's function for a membrane stretched over a two-dimensional domain  $B$  may be realized electrically as follows. Let a flat electrical insulator of the same shape as  $B$  be covered on both sides with tin foil, the two sides being connected along the boundary curve  $C$  so that electricity will flow freely from one side to the other. Then connect one pole of a battery to a point  $Q$  of  $B$  and the other pole to the corresponding point  $Q'$  on the other side. The electrical potential established over  $B$  will

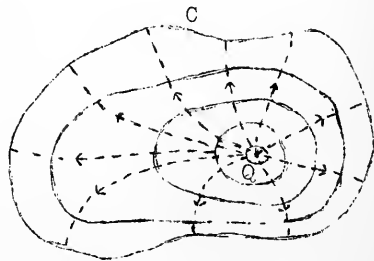


Fig. 10



then be exactly Green's function for a membrane which is fixed along C. The equipotential lines (C is, of course, an equipotential line and to it we may assign the value zero) are the level or contour lines of the surface  $u = K(x, y; \xi, \eta)$  where  $\xi$  and  $\eta$  are the coordinates of Q. Q, of course, is regarded as fixed in this discussion.

It is quite possible to obtain Green's function graphically by means of the following idea. Instead of employing a simple D.C. battery, we apply an A.C. voltage of audible frequencies. We can then find the equipotential lines,  $K = \text{constant}$ , in a stepwise manner. This is accomplished by using a sound receiving instrument, say a telephone, whose two terminals are connected to the tin foil on B through the points of a compass with a small opening. Keeping one point of the compass fixed, turn it around until the tone in the telephone disappears. When this occurs, the two compass points will be on the same equipotential line. By this process, in which we may take the compass opening as small as we please, we are able to find the system of equipotential lines and, thus, the level lines of K very quickly.

Another physical realization of Green's function for the above membrane is obtained if we consider the equation  $\nabla^2 u = 0$  as the equation describing the temperature of a homogeneous body as a function of position, if the heat flow is stationary. Let B be a flat, homogeneous, heat conducting body whose boundary C is kept at constant temperature, say  $u = 0$ . (E.g., this can be accomplished by immersing C in a large bath whose temperature is  $u = 0$ ). Now, at a point Q of B introduce a constant flow of heat into the body. The lines of constant temperature in the stationary flow pattern will be the level lines of Green's function for the membrane in the domain B. The singularity of Green's function will appear at the point Q where the heat flow is introduced.



## Part V. The Vibrating String and Membrane

### 1. General Theory of Natural Vibrations and Natural Modes.

#### Eigen-values and Eigen-functions.

We now leave the problem of equilibrium and take up that of motion. In particular, we will study the vibrations of continuous systems. All such vibrations can be described by a general mathematical theory analogous to the theory of normal coordinates discussed in connection with systems of a finite number of degrees of freedom. In the case of continuous systems, however, the theory is, in many respects, less abstract than was that for systems with a finite number of degrees of freedom. Furthermore, the general pattern for continuous systems is essentially the same for strings, membranes, cantilevers, plates, and many other similar systems.

The differential equation for the vibrations of any of the continuous systems mentioned above may be written in the general form

$$(1) \quad L[u] - c^2 u_{tt} = f(x, \dots; t)$$

where the dots in the right member indicate the possible presence of more than one space variable.  $L[u]$  is a linear differential expression which includes only the unknown function  $u$  and its derivatives with respect to the space variables  $x, \dots$ . Thus, for the string,

$$(2) \quad L[u] \equiv u_{xx} \quad ;$$

for the membrane

$$(3) \quad L[u] \equiv \nabla^2 u \quad ;$$

for the cantilever,

$$(4) \quad L[u] \equiv u_{xxxx} \quad ;$$

for the plate,

$$(5) \quad L[u] \equiv \nabla^2 \nabla^2 u \quad .$$



$c^2$  is a physical constant and  $f(x, \dots; t)$  is proportional to the density of the external force.

A specific vibrating system is characterized not alone by the differential equation (1), but also by boundary and initial conditions. For the most part, we shall deal with the simple homogeneous boundary condition  $u = 0$ . As our initial conditions we shall usually take the general non-homogeneous expressions

$$(6) \quad \begin{cases} u(x, \dots; 0) = \varphi(x, \dots) \\ u_t(x, \dots; 0) = \psi(x, \dots) \end{cases}$$

describing the prescribed initial state of the motion.

We consider, first, the problem of free vibrations. Mathematically, this means that we seek a function  $u$  which satisfies the homogeneous equation

$$(7) \quad L[u] - c^2 u_{tt} = 0,$$

a boundary condition, say  $u = 0$ , and the initial conditions (6). In particular, we seek solutions  $u$  of (7) which are "synchronized", that is, solutions which can be written as the product of a function of the space variables by a function of time, thus

$$(8) \quad u(x, \dots; t) = v(x, \dots) h(t).$$

Vibrations of this type are called the natural vibrations or natural modes. They behave as if the whole system essentially maintains its shape, while the amplitude of the shape, i.e., the factor  $h(t)$ , depends only on the time.

Substituting from (8) into (7) we get

$$\frac{L[v]}{c^2 v} = \frac{\ddot{h}}{h}.$$

Since the right side is a function of  $t$  alone and the left side is a function solely of the space variables, it follows that the two sides of the equation are equal to one and the same constant. Denoting this constant by  $-\lambda$  we obtain two differential equations





$$(9) \quad \ddot{h} + \lambda h = 0$$

$$(10) \quad L[v] + \lambda c^2 v = 0 \quad .$$

Thus the original equation (7) splits into two equations (9) and (10). The former has solutions  $h(t)$  of the form  $\cos \omega t$  and  $\sin \omega t$ , where  $\omega^2 = \lambda$ , while the latter, the so-called eigen-value equation, yields solutions which are functions of the space variables. Equation (10) determines the admissible values of  $\lambda$  the constant  $\lambda$  and, hence, the corresponding natural frequencies,  $\omega = \sqrt{\lambda}$ , of the system. This comes about as follows.

Together with equation (10) we must consider the homogeneous boundary condition imposed on (7), i.e.,  $u = 0$  on the boundary  $C$  of the domain of definition  $B$  of (7). By virtue of (8) this becomes a boundary condition for equation (10), i.e.,

$$(11) \quad v = 0 \quad \text{on } C \quad .$$

Now, the mathematical theory for the general case, as well as for explicitly treated cases, yields the following theorem, which we submit without proof:

There exists an infinite sequence  $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$  of values of  $\lambda$  for which the eigen-value equation (10) has a non-trivial solution which satisfies the boundary condition (11).

If this sequence of numbers is arranged in increasing order, say

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots \quad ,$$

then it turns out that  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Correspondingly, we have an infinite sequence of natural frequencies

$$\omega_1 \leq \omega_2 \leq \omega_3 \leq \dots \leq \omega_n \leq \dots \quad ,$$

where  $\omega_n^2 = \lambda_n$ .

The numbers  $\lambda_n$  are called the eigen-values or characteristic values of the boundary value problem given by (10) and (11). The corresponding solutions  $v_n(x, \dots)$  are the so-called eigen-functions or characteristic functions. It is



obvious that these solutions are not uniquely determined since they may be multiplied by any constant and still remain solutions. In part VI, this fact will enable us to "normalize" the eigen-functions. The eigen-functions  $v_n$  of the general problem possess many properties in common with the sine and cosine functions. We shall defer discussion of these properties to part VI and proceed here to discuss the second step in obtaining the solution of (1).

This step consists in superimposing solutions  $v_n \sin \omega_n t$  and  $v_n \cos \omega_n t$  of (7) to form the function

$$(12) \quad u(x, \dots; t) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) v_n(x, \dots)$$

which, of course, is also a solution of (7), and which, in addition, satisfies the boundary condition  $u = 0$  on  $C$  since the functions  $v_n$  satisfy the condition  $v_n = 0$  on  $C$ . The constants  $a_n$  and  $b_n$  may then be determined so that (12) will satisfy the initial condition (6). How this may be accomplished will be discussed in part VI.

As a final step, we consider the presence of an external force. This leads us back to the non-homogeneous equation (1). Since (1) is linear, we need find only one particular solution of it. This, combined with (12), then yields the general solution of (1). The particular solution may be obtained in either of two ways, which, however, are intrinsically related. The first way consists of expanding  $f(x, \dots; t)$  into a series in terms of the eigen-functions. Assuming the unknown function to possess a similar expansion we are able to obtain the coefficients of  $u$  from the differential equation. The accomplishment of these "generalized Fourier coefficients" will be discussed briefly in part VI. The second method is the general method of impulses which we shall consider in section 3.

Instead of continuing with these generalities at this point, we shall first consider a few important special examples which will, in fact, make more clear the need for more profound general mathematical theory.



## 2. The Vibrating String

The equation of motion of a simple elastic string is

$$(13) \quad u_{xx} - c^2 u_{tt} = f(x, t)$$

where  $c^2 = m/S$ ,  $S$  being the constant horizontal tension and  $m$  the constant mass density of the string. Let the string be fixed at both ends, i.e.,

$$(14) \quad u(0) = u(\ell) = 0$$

and have the initial state (6).

As in the previous section we consider first the homogeneous equation

$$(15) \quad u_{xx} - c^2 u_{tt} = 0$$

describing the free vibrations of the string. Seeking synchronized solutions of the form (8) we are led by (15) to the eigen-value problem

$$(16) \quad \begin{cases} v'' + \lambda c^2 v = 0 \\ v(0) = v(\ell) = 0 \end{cases}$$

and also to the differential equation (9) for  $h(t)$ .

Our task is to find the values of  $\lambda$ , the eigen-values, which yield the non-trivial solutions to the problem (16). It is clear that the only solutions of (16), within a constant factor, are

$$(17) \quad v_n(x) = \sin \frac{n\pi}{\ell} x, \quad n = 1, 2, 3, \dots$$

Hence

$$(18) \quad \lambda_n = \left( \frac{n\pi}{c\ell} \right)^2$$

and the natural frequencies are

$$(19) \quad \omega_n = \sqrt{\lambda_n} = \frac{n\pi}{c\ell}.$$

Also, from equation (9) we obtain



$$(20) \quad h(t) = \begin{cases} \sin \omega_n t \\ \cos \omega_n t \end{cases}.$$

By the principle of superposition,

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) v_n(x)$$

is a solution of (15). Clearly, it satisfies the boundary conditions (14). Furthermore, by properly choosing the  $a_n$  and  $b_n$ , it can be made to satisfy the initial conditions (6). We have

$$\begin{cases} u(x, 0) = \varphi(x) = \sum_{n=1}^{\infty} a_n v_n(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{\ell} x \\ u_t(x, 0) = \psi(x) = \sum_{n=1}^{\infty} \omega_n b_n v_n(x) = \sum_{n=1}^{\infty} \frac{n\pi}{c\ell} b_n \sin \frac{n\pi}{\ell} x \end{cases}.$$

Thus,  $a_n$  and  $\frac{n\pi}{c\ell} b_n$  are the Fourier coefficients of the pure sine series for  $\varphi(x)$  and  $\psi(x)$  respectively, i.e.,

$$(21) \quad a_n = \frac{2}{\ell} \int_0^{\ell} \varphi(\xi) \sin \frac{n\pi}{\ell} \xi d\xi$$

$$(21') \quad b_n = \frac{2c}{n\pi} \int_0^{\ell} \psi(\xi) \sin \frac{n\pi}{\ell} \xi d\xi.$$

Hence, the free vibrations of the string are given by

$$(22) \quad u(x, t) = \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{c\ell} t + b_n \sin \frac{n\pi}{c\ell} t) \sin \frac{n\pi}{\ell} x$$

where  $a_n$  and  $b_n$  are given by (21) and (21').

Before taking up the case of forced vibrations, we note the expressions for the kinetic and potential energies of the string. The kinetic energy is obviously

$$(23) \quad T = \frac{m}{2} \int_0^{\ell} u_t^2(x, t) dx,$$





while the potential energy  $V$  is proportional to the increase in length of the string, i.e.,

$$V = S \int_0^{\ell} (\sqrt{1 + u_x^2} - 1) dx \quad .$$

Recalling that we are always dealing with small oscillations, we may expand the radical in powers of  $u_x^2$  and neglect terms of second degree and higher. Thus

$$(24) \quad V = \frac{S}{2} \int_0^{\ell} u_x^2(x, t) dx \quad .$$

We may also write these two energy expressions in a manner which will remind us of the normal energy forms for systems with a finite number of degrees of freedom. For this purpose we write solution (22) in the form

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t) v_n(x)$$

where the  $q_n$  are easily identified by (22). Substituting this expression for  $u(x, t)$  into (23) we obtain, since

$$\int_0^{\ell} v_n^2 dx = \int_0^{\ell} \sin^2 \frac{n\pi}{\ell} x dx = \frac{\ell}{2}$$

and

$$\int_0^{\ell} v_n v_m dx = \int_0^{\ell} \sin \frac{n\pi}{\ell} x \sin \frac{m\pi}{\ell} x dx = 0 \quad ,$$

the quadratic form

$$(23') \quad T = \frac{m\ell}{4} \sum_{n=1}^{\infty} \dot{q}_n^2 \quad .$$

Likewise, integrating by parts and using (16), we have



$$\begin{aligned} \int_0^{\ell} (v_n')^2 dx &= v_n v_n' \Big|_0^{\ell} - \int_0^{\ell} v_n v_n'' dx \\ &= - \int_0^{\ell} v_n v_n'' dx = \lambda_n c^2 \int_0^{\ell} v_n^2 dx \end{aligned}$$

or

$$\int_0^{\ell} (v_n')^2 dx = \frac{\ell c^2}{2} \lambda_n.$$

Hence

$$(24') \quad V = \frac{S \ell c^2}{4} \sum_{n=1}^{\infty} \lambda_n a_n^2.$$

We have seen in Chapter One that finite sums similar in form to the infinite sums (23') and (24') represent the kinetic and potential energies of systems with a finite number of degrees of freedom. The generalization of the finite sums to the infinite may seem to be but a slight one. However, mathematically, such infinite sums may present something entirely new, and more subtle methods must be developed for handling them.

In passing we might also note that the reduced energy forms  $\bar{T}$  and  $\bar{V}$ , corresponding to  $T$  and  $V$ , are

$$(25) \quad \bar{T} = \frac{m \ell}{4} \sum_{n=1}^{\infty} a_n^2$$

$$(26) \quad \bar{V} = \frac{S \ell c^2}{4} \sum_{n=1}^{\infty} \lambda_n a_n^2$$

obtained by setting  $u(x,t) = v(x)e^{i\omega t}$  where  $v(x) = \sum_{n=1}^{\infty} a_n v_n(x)$ .

We return now to the problem of forced vibrations. If, in (13),  $f(x,t)$  is a sufficiently smooth function which vanishes at  $x = 0$  and  $x = \ell$ , then by the theory of Fourier series,

$$(27) \quad f(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi}{\ell} x^*,$$

---

\* If it is not true that  $f(x,t)$  vanishes at  $x = 0$  and  $x = \ell$ , then the right side of (27) will represent  $f(x,t)$  only in  $0 < x < \ell$  and will be zero at  $x = 0$  and  $x = \ell$ . With this in mind, we may suppose  $f(x,t)$  to be represented by (27) even if the condition of vanishing is not satisfied.



where

$$(28) \quad A_n(t) = \frac{2}{\ell} \int_0^{\ell} f(\xi, t) \sin \frac{n\pi}{\ell} \xi d\xi \quad .$$

We may then suppose that a particular solution of (13) has the form

$$(29) \quad u(x, t) = \sum_{n=1}^{\infty} U_n(t) \sin \frac{n\pi}{\ell} x$$

where the  $U_n(t)$  are to be determined. Substituting from (27) and (29) into (13) we obtain a system of ordinary differential equations

$$(30) \quad x^2 \ddot{U}_n(t) + \lambda_n c^2 U_n(t) = -A_n(t) \quad , \quad (n = 1, 2, \dots) \quad .$$

In order to obtain a particular solution of (30) we impose the initial conditions  $u(x, 0) = u_t(x, 0) = 0$ , whence

$$(31) \quad U_n(0) = \dot{U}_n(0) = 0 \quad , \quad (n = 1, 2, \dots) \quad .$$

Thus the problem of finding a particular solution of (13) has been reduced to the problem of finding the solution to the initial value problem for a normal system of ordinary differential equations. This problem was considered in Chapter One.

To obtain the complete solution of the boundary and initial value problem for the motion of a string under a prescribed external force, we need only superimpose (29) and the solution (22) of the free vibration problem. That the boundary and initial conditions are satisfied may be easily verified.

Instead of actually finding a particular solution in this manner, we may employ the so-called impulse-method. In the following section we carry out this method and investigate its physical significance.



### 3. The Impulse Method

The principle of this method is suggested, as was that of Green's function, by a physical idea. We have, heretofore, considered the external force function  $f(x, \dots; t)$  to be continuous in the time  $t$ , that is, the impressed force has been a steady one. We now replace this continuously acting force by one which acts in a sequence of discrete impulses. If the number of these impulse forces in a given time is permitted to increase while, simultaneously, the intensity of each impulse is allowed to decrease, then, in the limit, we have again the steady force with continuous density function  $f(x, \dots; t)$ .

We investigate, first, the effect on a system of a single impulse applied at a time  $t = \tau$ . Such an impulse may be looked upon as the limiting case of a force whose density  $\varphi(x, \dots; t)$  is very large in a small time interval  $(\tau - \varepsilon \leq t \leq \tau + \varepsilon)$  about  $t = \tau$  and equal to zero outside the interval.

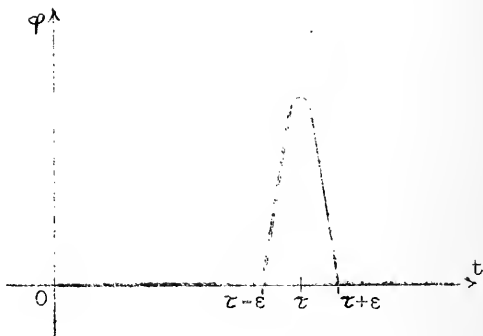


Fig. 1

If the value of  $\varphi$  in the interval increases at a proper rate as  $\varepsilon \rightarrow 0$ , then the limiting force density will be a given function  $f(x, \dots; t)$  for  $t = \tau$ , that is,

$$(32) \quad \lim_{\varepsilon \rightarrow 0} \int_{\tau - \varepsilon}^{\tau + \varepsilon} \varphi(x, \dots; t) dt = f(x, \dots; \tau) \quad .$$

The physical system, subjected to force of density  $\varphi(x, \dots; t)$ , will have free vibrations for  $t < \tau - \varepsilon$  and for  $t > \tau + \varepsilon$  while during the time interval  $(\tau - \varepsilon \leq t \leq \tau + \varepsilon)$  forced vibrations will take place. The differential equation of motion is, thus,

$$(7) \quad L[u] - c^2 u_{tt} = 0 \quad \text{for} \quad |t - \tau| > \varepsilon$$

and





$$(33) \quad L[u] - c^2 u_{tt} = \varphi(x, \dots; t) \quad \text{for} \quad |t - \tau| < \varepsilon.$$

Integrating (33) over the small time interval, we have

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} L[u] \, d\tau - c^2 \int_{\tau-\varepsilon}^{\tau+\varepsilon} u_{tt} \, dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon} \varphi \, dt.$$

Letting  $\varepsilon \rightarrow 0$ , we get

$$(34) \quad \lim_{\varepsilon \rightarrow 0} [u_t(x, \dots; \tau+\varepsilon) - u_t(x, \dots; \tau-\varepsilon)] = -c^2 f(x, \dots; \tau),$$

since, under such an impulse, we should expect that the resulting motion will be a free vibration for  $t < \tau$  for which the deflection  $u$  and its space derivatives are continuous for all  $t$ , but whose velocity at  $t = \tau$  suffers a jump discontinuity of amount proportional to  $f(x, \dots; \tau)$ .

We now assume the system whose motion is given by

$$(1) \quad L[u] - c^2 u_{tt} = f(x, \dots; t)$$

to be initially at rest, i.e.,

$$(35) \quad \begin{cases} u(x, \dots; 0) = 0 \\ u_t(x, \dots; 0) = 0 \end{cases}.$$

If the continuous external force in (1) is replaced by a single impulse acting at time  $t = \tau$ , then the resulting motion is given by

$$(36) \quad u = H(x, \dots; t; \tau)$$

where  $H(x, \dots; t; \tau)$  satisfies the homogeneous equation (7) and is such that  $H = H_t = 0$  for  $t < \tau$ . Since  $H$  is continuous this means that  $H = 0$  also for  $t = \tau$ , while, by (34),  $H_t = -c^2 f(x, \dots; \tau)$  for  $t = \tau$ . In other words,  $H$  is characterized as the solution of the following initial value problem:



$$(37) \quad \begin{cases} L[H] - c^2 H_{tt} = 0 \\ H(x, \dots; \tau; \tau) = 0 \\ H_t(x, \dots; \tau; \tau) = -c^{-2} f(x, \dots; \tau) \end{cases} .$$

It is plausible that, for a continuously acting force, the resulting motion of the system described by (1) and (35) will, up to a time  $t$ , be a superposition of the effects of impulses acting at every time  $\tau \leq t$ , namely,

$$(38) \quad u(x, \dots; t) = \int_0^t H(x, \dots; t; \tau) d\tau .$$

So far our reasoning has been merely heuristic and not of the nature of a mathematical proof. However, by discarding these plausibility considerations and by simply defining  $H$  by (37), an initial value problem which can be solved by previous methods, it is a simple mathematical exercise to show that (38) is the required solution of equation (1) with the initial state (35). To show this, we first observe in (38) that  $u = 0$  for  $t = 0$ . To find  $u_t$ , we differentiate (38), obtaining

$$u_t = H(x, \dots; t; t) + \int_0^t H_t(x, \dots; t; \tau) d\tau$$

which vanishes for  $t = 0$ , since  $H = 0$  for  $t = \tau$ . Thus the initial conditions (35) are satisfied. To verify that (38) satisfies the differential equation (1), we find

$$\begin{aligned} u_{tt} &= H_t(x, \dots; t; t) + \int_0^t H_{tt} d\tau \\ &= -c^{-2} f(x, \dots; t) + \int_0^t H_{tt} d\tau \end{aligned}$$

by virtue of the second initial condition of (37). Also,

$$L[u] = \int_0^t L[H] d\tau .$$



Therefore, by the differential equation of (37),

$$\begin{aligned} L[u] - c^2 u_{tt} &= \int_0^t (L[H] - c^2 H_{tt}) d\tau + f(x, \dots; t) \\ &= f(x, \dots; t) \end{aligned}$$

The impulse method yields a particular solution (38) of equation (1) which satisfies the initial conditions (35). The solution may also be made to satisfy the homogeneous boundary condition  $u = 0$  on the boundary  $C$  of the domain of the system. This can be accomplished by letting  $H(x, \dots; t; \tau)$  satisfy the boundary condition  $H = 0$  on  $C$ , as well as the homogeneous initial conditions. For, if  $H = 0$  on  $C$ , we see immediately by (38) that  $u = 0$  on  $C$ . Hence, the solution of the homogeneous boundary and initial value problem of the forced vibrations given by (1) is the sum of the solution of the homogeneous equation satisfying the boundary and initial conditions and the particular solution (38) of the non-homogeneous equation satisfying the boundary condition.

We may, as an illustration, apply the method of impulses to the string problem of the last section. To determine  $H(x; t; \tau)$  we recall that the general solution of

$$u_{xx} - c^2 u_{tt} = 0$$

was given by (22) which, for convenience, we rewrite in the form

$$(39) \quad H(x; t; \tau) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{c\ell}(t-\tau) + b_n \sin \frac{n\pi}{c\ell}(t-\tau) \right) \sin \frac{n\pi}{\ell} x.$$

Setting

$$\begin{cases} a_n = 0 \\ b_n = -\frac{2}{n\pi c} \int_0^{\ell} f(\xi, t) \sin \frac{n\pi}{\ell} \xi d\xi \end{cases},$$

we see that the initial conditions of (37) are satisfied as well as the boundary condition  $H(0; t; \tau) = 0$ . The required solution is, then,



$$\int_0^t H(x;t;\tau) d\tau \quad .$$

Exercise: Carry out in detail the procedure above for the string and obtain an explicit solution. Also, find the particular solution by the method of section 2 by solving the equations (30) by one of the methods of Chapter One. Compare the two results and verify that the two methods are essentially the same.

#### 4. The Non-homogeneous String

Whenever we have discussed the problem of the elastic string we have assumed, for simplicity, that the mass density  $m$  was constant. We now consider the more general case where  $m$  is a positive continuous function  $m = \rho(x)$  along the length of the string. The equation of free vibrations in this case is

$$(40) \quad u_{xx} - \rho(x)u_{tt} = 0 \quad * .$$

Suppose the string to be of unit length with ends fixed at  $x = 0$  and  $x = 1$ , and to have the initial state

$$(41) \quad \begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} .$$

If we attempt to find synchronous solutions  $u = v(x) h(t)$ , we obtain sine and cosine solutions for  $h(t)$ , while for  $v(x)$  we have to solve the eigen-value problem

$$(42) \quad \begin{cases} v'' + \lambda \rho(x) v = 0 \\ v(0) = v(1) = 0 \end{cases} .$$

The differential equation involved here is not of the same simple type as that encountered in the case of the homogeneous string. It is the so-called Sturm-Liouville equation, of which equation (16) is a special case. Extensive studies of this equation have

---

\* For convenience we let  $S = 1$ .





been made revealing that only in very special cases are the solutions of the boundary value problem (42) given in terms of elementary functions.

Nevertheless, the mathematical theory demonstrates that there exists an infinite sequence of eigen-values  $\lambda_n$ , which tends to infinity as  $n$  tends to infinity, and a corresponding sequence of eigen-functions  $v_n(x)$ . The existence of such solutions, however, is not enough for us. We are interested mainly in methods for obtaining them, at least approximately. Such a method will be considered in Chapter Three.

For the moment, suppose that the  $\lambda_n$  and  $v_n$  are known. The solution to (40) is then given by the superposition

$$(43) \quad u(x, t) = \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t) v_n(x) \quad ,$$

where the  $\omega_n = \sqrt{\lambda_n}$  are the natural frequencies of the string. The  $a_n$  and  $b_n$  must be determined so that the initial conditions are satisfied. Hence

$$(44) \quad \varphi(x) = \sum_{n=1}^{\infty} a_n v_n(x)$$

$$(45) \quad \psi(x) = \sum_{n=1}^{\infty} \omega_n b_n v_n(x) \quad .$$

We are thus confronted with another problem. In general, the  $v_n(x)$  will not be functions of the form  $\sin(nkx)$  or  $\cos(nkx)$ , ( $k = \text{const.}$ ), and, therefore, the ordinary methods of Fourier analysis cannot be applied. However, in Part VI we shall see that the eigen-functions possess properties in common with  $\sin(nkx)$  and  $\cos(nkx)$ , properties which make possible the development of "generalized" Fourier expansions.

Exercise: Solve the problem of the rotating string given by

$$(46) \quad S u_{xx} - m u_{tt} + k u = 0 \quad ,$$

where  $k$  is constant proportional to the angular velocity. Let the string have unit length and be fixed at  $x = 0$  and  $x = 1$ .



## 5. Strings with Other Boundary Conditions

Another problem which cannot be solved by the ordinary Fourier expansion methods is the following. Consider a homogeneous string fixed at one end, say  $x = 0$ , and with its other end free to move along the line  $x = \pi$ . To investigate the free motion of such a string we assume, for simplicity, that  $S = m = 1$ . The differential equation is then

$$(47) \quad u_{xx} - u_{tt} = 0$$

and the boundary conditions are

$$(48) \quad u(0) = 0 \quad \text{and} \quad u'(\pi) = 0$$

Attempting to find synchronous solutions  $u = v(x)h(t)$  we obtain for  $v(x)$  the eigen-value problem

$$(49) \quad \begin{cases} v'' + \lambda v = 0 \\ v(0) = v'(\pi) = 0 \end{cases} .$$

Obviously,  $\sin \sqrt{\lambda} x$  satisfies the differential equation and the first boundary condition. In order that the condition at  $x = \pi$  be satisfied we must have  $\cos \sqrt{\lambda} \pi = 0$ , or

$$(50) \quad \lambda_n = \frac{1}{4}(2n - 1)^2, \quad (n = 1, 2, \dots)$$

whence

$$(51) \quad v_n(x) = \sin \frac{2n-1}{2} x .$$

The eigen-functions are, thus, elementary functions, but they are not of the type that makes possible the use of ordinary Fourier methods to satisfy the initial conditions

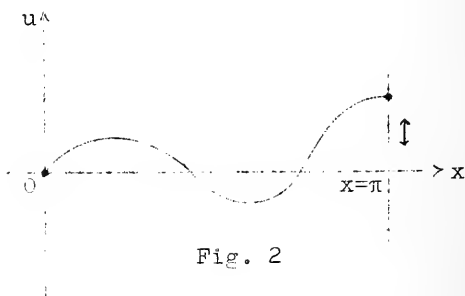


Fig. 2



$$\begin{cases} u(x, 0) = q(x) \\ u_t(x, 0) = \psi(x) \end{cases}.$$

A variation of this problem may be obtained by considering the right endpoint of the string not as moving freely along the line  $x = \pi$  but constrained by some elastic force. The second initial condition then becomes

$$(52) \quad u'(\pi) + ku(\pi) = 0, \quad *$$

where  $k$  is a positive constant. In order that  $\sin \sqrt{\lambda}x$  satisfy (52) we must choose  $\lambda$  such that

$$\sqrt{\lambda} \cos \sqrt{\lambda} \pi + k \sin \sqrt{\lambda} \pi = 0$$

or

$$\tan \sqrt{\lambda} \pi = -\frac{\sqrt{\lambda}}{k}.$$

Letting  $\xi = \sqrt{\lambda} \pi$ , we have the transcendental equation

$$(53) \quad \tan \xi = -\frac{1}{k\pi} \xi$$

for  $\xi$ , and, consequently, for  $\lambda$ . This equation has infinitely many roots  $\xi_1, \xi_2, \dots, \xi_n, \dots$  which are such that  $\xi_n \rightarrow \infty$  as  $n \rightarrow \infty$  if we arrange them

in increasing order. These

roots may be found graphically

by determining the inter-

sections of the tangent curves

$y = \tan \xi$  with the line

$y = (-1/k\pi)\xi$ . (See figure

3). The eigen-values are

then equal to  $\xi_n^2/\pi^2$  and the

corresponding eigen-functions

are

$$(54) \quad v_n(x) = \sin \sqrt{\lambda_n} x.$$

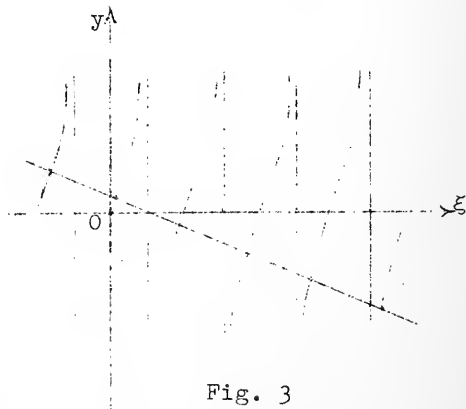


Fig. 3

\* The question as to whether such conditions are really the ones which describe the physical situation will be answered in a general manner in Chapter Three.



## 6. The Square Membrane

Let an elastic membrane be stretched over the square domain

$$\begin{cases} 0 \leq x \leq \pi \\ 0 \leq y \leq \pi \end{cases}$$

of the  $x, y$ -plane. For convenience, suppose that the horizontal tension  $S$  and the mass density  $m$  are both unity. If we assume the membrane to be fixed along the sides of the

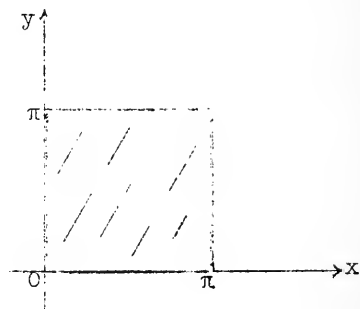


Fig. 4

square, the free motion of the membrane is given by the following boundary and initial value problem:

$$(55) \quad \begin{cases} \nabla^2 u - u_{tt} = 0 \\ u = 0 \quad \text{on the sides of the square,} \\ u(x, y; 0) = \varphi(x, y) \\ u_t(x, y; 0) = \psi(x, y) \end{cases}$$

Attempting to find solutions of the form  $u = v(x, y) h(t)$ , we arrive at the following eigen-value problem:

$$(56) \quad \begin{cases} \nabla^2 v + \lambda v = 0 \\ v = 0 \quad \text{on the sides of the square} \end{cases}$$

The function  $h(t)$  is, again, of the form  $\cos \omega_n t$  or  $\sin \omega_n t$ . An examination of (56) reveals that it is satisfied by the product

$$\sin nx \sin my \quad (n, m \text{ integers})$$

if  $\lambda = n^2 + m^2$ . Thus, the solution to (55) may be written

$$(57) \quad u(x, y; t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \sqrt{n^2 + m^2} t + b_{nm} \sin \sqrt{n^2 + m^2} t) \sin nx \sin my$$





where

$$\begin{cases} a_{nm} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} \varphi(\xi, \eta) \sin n\xi \sin m\eta \, d\xi \, d\eta \\ b_{nm} = \frac{4}{\pi^2 \sqrt{n^2 + m^2}} \int_0^{\pi} \int_0^{\pi} \psi(\xi, \eta) \sin n\xi \sin m\eta \, d\xi \, d\eta \end{cases}$$

are the Fourier coefficients of  $\varphi$  and  $\psi$  respectively.

The eigen-values, in this case, are

$$(58) \quad \lambda_{nm} = n^2 + m^2$$

and the eigen-functions

$$(59) \quad v_n(x, y) = \sin nx \sin my \quad .$$

Despite the fact that the  $\lambda_{nm}$  are characterized by two subscripts  $n$  and  $m$ , it is possible to arrange them in a non-decreasing sequence

$$\lambda^{(1)} \leq \lambda^{(2)} \leq \lambda^{(3)} \leq \dots$$

where

$$\lambda^{(1)} = \lambda_{11} = 2$$

$$\lambda^{(2)} = \lambda_{12} = 5$$

$$\lambda^{(3)} = \lambda_{21} = 5$$

$$\lambda^{(4)} = \lambda_{22} = 8$$

$$\lambda^{(5)} = \lambda_{13} = 10$$

$$\lambda^{(6)} = \lambda_{31} = 10$$

etc.

Obviously, we must consider  $\lambda_{ij}$  and  $\lambda_{ji}$  as two different eigen-values, for the corresponding amplitudes of the natural modes  $v_{ij}$  and  $v_{ji}$  are certainly not the same. We have thus encountered a remarkable fact which we have not come across before, i.e.,



to one and the same numerical value of the natural frequency there may correspond more than one natural mode. Eigen-values of this sort are called many-valued eigen-values. The question as to the many-valuedness of any particular eigen-value is answered by the number-theoretic problem of determining the number of ways  $\lambda_{nm}$  may be expressed as the sum of the squares of two positive integers.

Before leaving the square membrane, we mention briefly a noteworthy phenomenon which occurs in vibrating systems. The motion of an elastic string was given by a superposition of natural modes  $v_n(x)e^{i\omega_n t}$ . While the string is vibrating in such a natural mode, there may be points of the string which remain motionless. If such points exist they will be given by the zeros of the eigen-function  $v_n(x)$ . They are called the nodal points of the natural mode  $v_n(x)e^{i\omega_n t}$ . In the case of the membrane, the same phenomenon occurs. However, the vanishing of  $v_n(x,y)$  is not restricted to a point but may also occur along lines or curves in the membrane. Such lines are called nodal lines of the natural mode  $v_n(x,y)e^{i\omega_n t}$ .

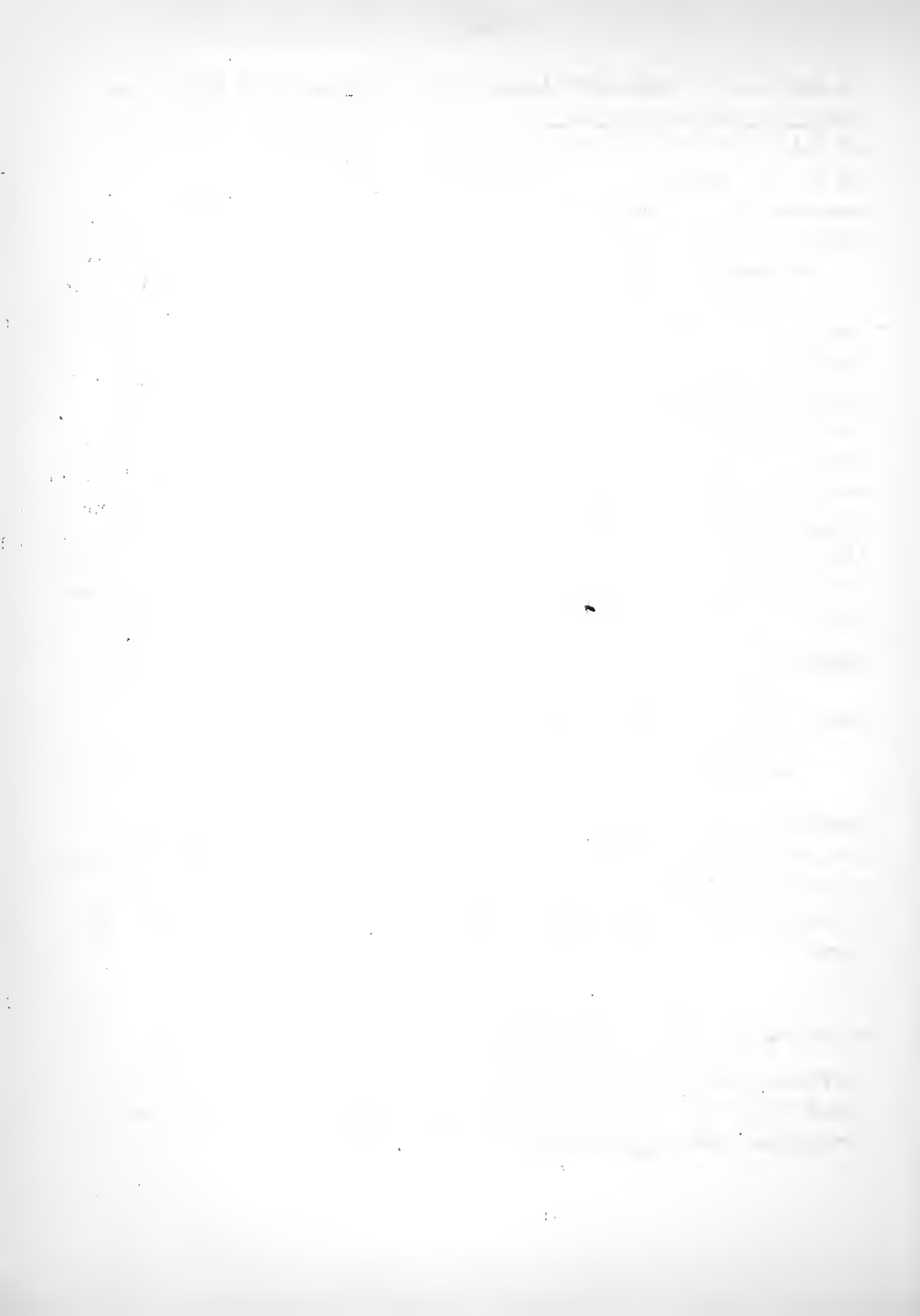
For the square membrane the nodal lines for the natural modes corresponding to  $\sin nx \sin my$  are the lines

$$nx = \nu\pi, \quad my = \mu\pi, \quad (\nu = 0, 1, \dots, n-1; \mu = 0, 1, \dots, m-1).$$

However, since many-valued frequencies occur in this case, linear combinations of the different natural modes corresponding to the same natural frequency furnish other nodal lines with respect to that frequency. Thus, the zeros of functions of the form

$$\alpha \sin nx \sin my + \beta \sin mx \sin ny, \quad ,$$

where  $\alpha$  and  $\beta$  are real constants, give us the nodal lines corresponding to  $\omega_n = \sqrt{n^2 + m^2}$ . Figure 5 shows some of the nodal lines for different natural frequencies of the square membrane. Here  $v_{nm}$  denotes  $\sin nx \sin my$ .



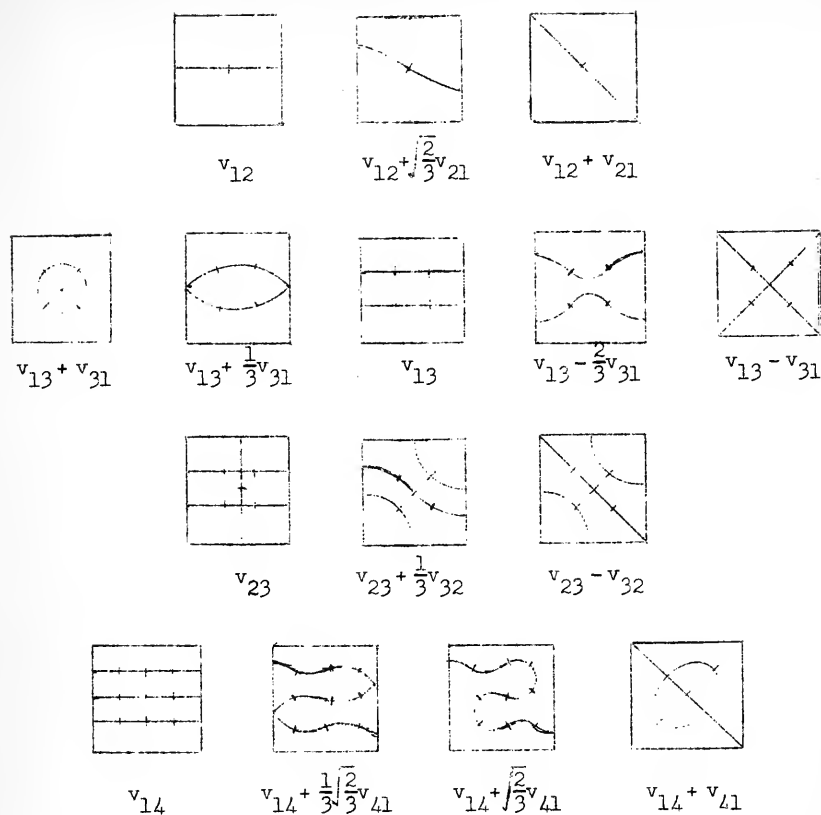


Fig. 5

Finally, a word may be said about the energies of a membrane. Suppose the membrane to be stretched over a general domain  $B$  and let the constants  $m$  and  $S$  be the mass density and horizontal tension respectively. Then, as usual, the kinetic energy is

$$(60) \quad T = \frac{m}{2} \iint_B u_{tt}^2 \, dx \, dy$$

while the potential energy is proportional to the increase in area, i.e.,



$$V = S \iint_B (\sqrt{1 + u_x^2 + u_y^2} - 1) dx dy .$$

Recalling that we are dealing with small oscillations only, we find, by expanding the radical and neglecting high powers, that

$$(61) \quad V = \frac{S}{2} \iint_B (u_x^2 + u_y^2) dx dy .$$

### 7. The Circular Membrane

As a final example of the application of the eigen-value theory, we consider an elastic membrane stretched over the unit circle  $x^2 + y^2 \leq 1$ . Letting  $m = S = 1$ , the boundary and initial value problem for the free motion of the membrane fixed along the boundary is given by

$$(62) \quad \begin{cases} \nabla^2 u - u_{tt} = 0 \\ u = 0 \text{ along } x^2 + y^2 = 1 \\ u(x, y; 0) = \varphi(x, y) \\ u_t(x, y; 0) = \psi(x, y) \end{cases} .$$

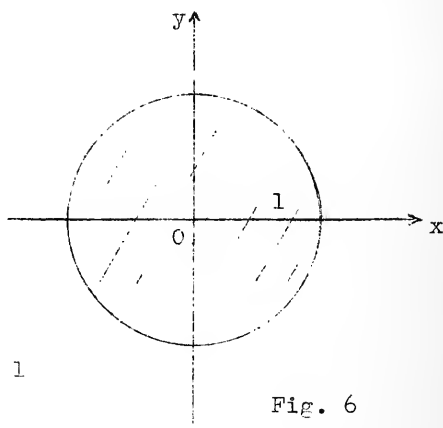


Fig. 6

The introduction of synchronous motions  $u = v(x, y) h(t)$  leads to the eigen-value problem

$$(63) \quad \begin{cases} \nabla^2 v + \lambda v = 0 \\ v = 0 \text{ on } x^2 + y^2 = 1 \end{cases} .$$

The elementary trigonometric functions do not furnish the desired eigen-functions in this case. To solve (63) we resort to polar coordinates by making the transformation





$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} .$$

Equations (63) are transformed into

$$(64) \quad \begin{cases} v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} + \lambda v = 0 \\ v(1, \theta) = 0 \end{cases} .$$

Problem (64) may seem at first glance to be even more imposing than (63). It can, however, be reduced to a problem involving two ordinary differential equations by attempting to find solutions of the form

$$(65) \quad v(r, \theta) = w(r) g(\theta) .$$

Substituting from (65) into (64), we get

$$\frac{r^2(w''(r) + \frac{1}{r} w'(r) + \lambda w(r))}{w(r)} = - \frac{g''(\theta)}{g(\theta)} = c = \text{const.},$$

or

$$(66) \quad r^2 w'' + r w' + (\lambda - c) w = 0$$

and

$$(67) \quad g'' + c g = 0 .$$

Since the function  $v(r, \theta)$  must be periodic\* (with period  $2\pi$ ) in  $\theta$ , the same must be true of  $g(\theta)$ ; thus  $c = n^2$ , where  $n$  is any non-negative integer. Hence

$$(68) \quad g(\theta) = \alpha \cos n\theta + \beta \sin n\theta$$

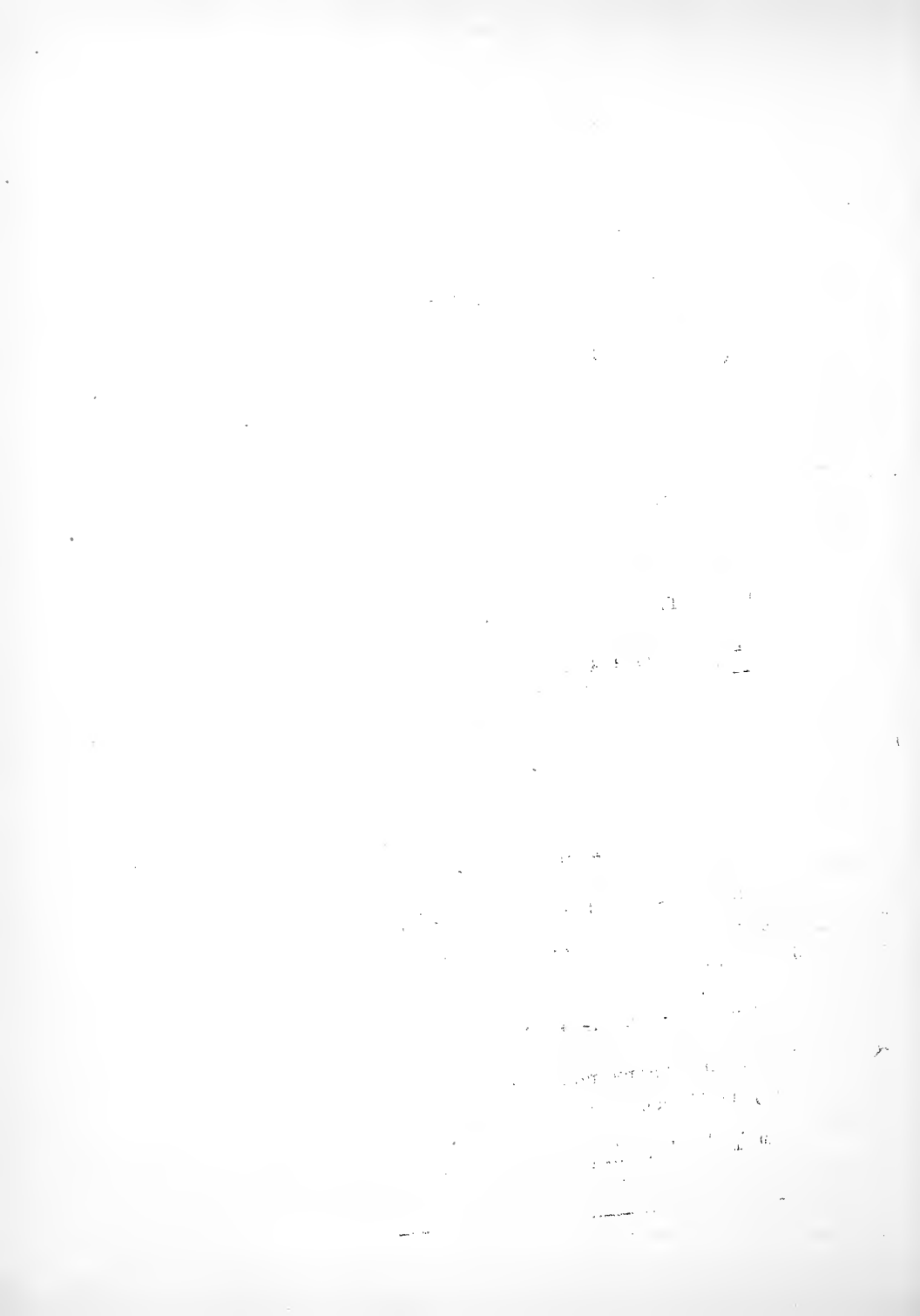
where  $\alpha$  and  $\beta$  are arbitrary real constants.

For  $c = n^2$ , (66) becomes

$$(69) \quad w'' + \frac{1}{r} w' + \frac{1}{r^2} (\lambda - n^2) w = 0$$

---

\* For  $v(r, \theta)$  gives the amplitude of the deflection  $u$  at the point  $(r, \theta)$  of the circle and, since  $(r, \theta)$  and  $(r, \theta + 2\pi)$  represent the same point,  $v(r, \theta)$  must be the same as  $v(r, \theta + 2\pi)$ .



with the boundary condition  $w(1) = 0$ . We may transform the latter equation again by letting  $\lambda = k^2$  and  $kr = \rho$ , where  $k$  is a real number. Then  $w(r) \equiv z(\rho)$  and

$$(70) \quad z'' + \frac{1}{\rho} z' + (1 - \frac{n^2}{\rho^2})z = 0$$

with the boundary condition  $z(k) = 0$ . We notice, however, that (70) is a second order equation and should, therefore, have imposed upon it another boundary condition. We do this by requiring that the solution of (70) for which  $z(k) = 0$  be continuous for  $\rho = 0$ . That this is actually an additional condition is indicated by the fact that the differential equation (70) possesses a strong singularity for  $\rho = 0$ .

Equation (70) is the so-called Bessel differential equation and the solution which remains continuous at  $\rho = 0$  is called the Bessel function of the  $n^{\text{th}}$  order,  $J_n(\rho)$ . Without going into the details of the theory of Bessel functions\*, we merely indicate that the required solution of (70) may be found by assuming the solution  $z(\rho)$  to possess a power series development

$$z(\rho) = \sum_{v=0}^{\infty} a_v \rho^v \quad .$$

Substituting this into (70), we find

$$(71) \quad z(\rho) = J_n(\rho) = \frac{\rho^n}{2^n n!} \left[ 1 - \frac{\rho^2}{2(2n+2)} + \frac{\rho^4}{2 \cdot 4(2n+2)(2n+4)} - \dots \right] \quad .$$

Returning to the boundary value problem for (70) we must still satisfy the condition  $z(k) = 0$ , that is

$$J_n(k) = 0 \quad .$$

Thus, the eigen-values  $\lambda = k^2$  are the squares of the roots of the Bessel functions. As a matter of fact,  $J_n(\rho)$  has an

---

\* See Karman and Biot, Chap. II, for a discussion of Bessel functions.



infinite number of discrete real roots  $k_{n\nu}$  ( $\nu=1,2,\dots$ ). (See figure 7 for a graph of the 0<sup>th</sup> order Bessel function  $J_0(\rho)$ ). Hence, the eigen-values of (63) are

$$(72) \quad \lambda_{n\nu} = k_{n\nu}^2$$

( $n=0,1,2,\dots; \nu=1,2,3,\dots$ ) and the corresponding eigen-functions are

$$(73) \quad v_n(r, \theta) = J_n(k_{n\nu}r) (\alpha \cos n\theta + \beta \sin n\theta) \quad .$$

The fact that  $\alpha$  and  $\beta$  are arbitrary suggests that the eigen-values are at least double-values, for  $\lambda_{n\nu}$  corresponds to the two linearly independent eigen-functions  $J_n \cos n\theta$  and  $J_n \sin n\theta$ . The corresponding natural modes of the motion of the membrane are

$$(74) \quad J_n(k_{n\nu}r) (\alpha \cos n\theta + \beta \sin n\theta) (a \sin k_{n\nu}t + b \cos k_{n\nu}t)$$

where, of course, the  $a$  and  $b$  are determined by the initial state of the motion. We shall return in part VI to the question of the determination of the  $a$  and  $b$  by the expansion of arbitrary functions in terms of the eigen-functions (73).

## Part VI. A Summary of the General Theory of Eigen-values

At the beginning of Part V we say that eigen-value problems arise when we consider the motion of a continuous physical system given by a partial differential equation of the form

$$(1) \quad L[u] - \rho(x, \dots)u_{tt} = f(x, \dots; t)$$

together with prescribed boundary and initial conditions for the function  $u(x, \dots; t)$ . The equation as we have written it here is somewhat more general than that we considered, in that we

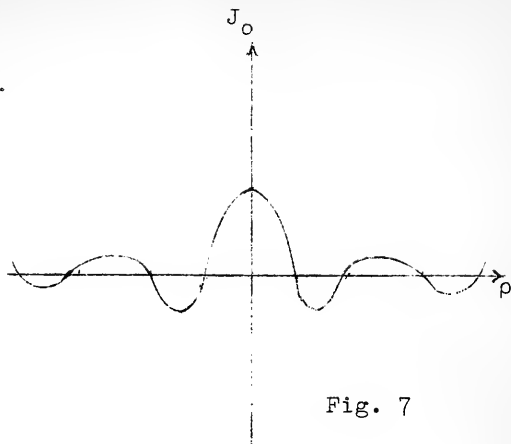


Fig. 7



treated the special case where the positive mass density function  $\rho(x, \dots)$  was a constant  $c^2$ . Our general procedure consisted in attempting to find synchronous solutions  $u = v(x, \dots) h(t)$  of the homogeneous equation associated with equation (1). This led to the two ordinary differential equations

$$(2) \quad \ddot{h} + \lambda h = 0$$

and

$$(3) \quad L[v] + \lambda \rho v = 0 \quad ,$$

the first of which yielded solutions of the form  $e^{i\sqrt{\lambda}t}$ , while the second furnished the eigen-value problem. It is on the latter that we focus our attention.

The boundary conditions (in the problems we consider) imposed on  $u(x, \dots; t)$  are always homogeneous. Thus, for the string with fixed endpoints or the membrane fixed along a curve in a plane we have  $u = 0$  on the boundary  $C$  of the domain  $B$ . For the clamped cantilever or plate, both  $u$  and its first derivatives are required to vanish on  $C$ . These conditions are expressed mathematically by

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } C \quad ,$$

where  $\frac{\partial u}{\partial n}$  is the derivative of  $u$  in the direction of the outward normal to  $C$ . Other typical homogeneous boundary conditions are:  $\frac{\partial u}{\partial n} = 0$  on  $C$ , if the string or membrane is permitted free motion at the boundary;  $\frac{\partial u}{\partial n} + \sigma u = 0$  on  $C$ , if the string or membrane is in motion at the boundary but is constrained there by an elastic force proportional to the positive function  $\sigma(x, \dots)$ . Also, it is possible to have several of the above homogeneous conditions imposed simultaneously at various parts of the boundary  $C$ , as, e.g., in section 4 of part V.

We readily see that the homogeneous boundary conditions imposed on  $u$  carry over to the function  $v$  in equation (3). Thus the eigen-value problem which presents itself is that of finding those values of  $\lambda$  for which there exist non-trivial solutions of (3) satisfying the boundary conditions. We have already mentioned and have, in fact, actually verified in special cases, that there exists an infinite sequence





$$(4) \quad \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots$$

( $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ ) of values of  $\lambda$  to which correspond non-trivial solutions

$$(5) \quad v_1, v_2, v_3, \dots, v_n, \dots$$

of the boundary value problem. (The proof of this fact for the general equation (3) is omitted here, for to present it would take us far afield from our original aims in this course.)

Furthermore, we state that in the problems which we are treating here the  $\lambda_n$  are non-negative\* and  $\omega_n = \sqrt{\lambda_n}$  is the natural frequency of the  $n^{\text{th}}$  natural mode  $v_n e^{i\omega_n t}$  of the system.

### 1. Orthogonality and Completeness of the Eigen-functions

We have already mentioned that the eigen-function  $v_n(x, \dots)$  possess many properties in common with the trigonometric functions  $\sin nx$  and  $\cos nx$ , where  $n = 0, 1, 2, \dots$ . One of these is the so-called orthogonality property which, for the sine and cosine functions, is expressed by the relations

$$\int_0^\pi \sin n\xi \sin m\xi d\xi = \int_0^\pi \cos n\xi \cos m\xi d\xi = \begin{cases} 0 & \text{if } n \neq m \\ \frac{\pi}{2} & \text{if } n = m \end{cases}.$$

The analogous relations for the general eigen-functions are

$$(6) \quad \int_B \dots \int v_n v_m d\xi \dots = \begin{cases} 0 & \text{if } n \neq m \\ c_n & \text{if } n = m \end{cases}.$$

The relations (6) are more general in form, in that the function  $\rho(x, \dots)$  appears.

---

\* This is due to the fact that the potential energy of each of the systems we are treating is a positive definite quadratic expression.



There is some inconvenience in having

$$(6') \quad \int_B \rho v_n^2 d\xi \quad \text{---} = c_n \quad .$$

That is, we prefer the right side to be unity. To achieve this we "normalize" the eigen-functions. For example, to normalize  $\sin nx$  we multiply it by a factor  $\mu_n$  such that

$$\mu_n^2 \int_0^n \sin^2 n\xi d\xi = 1 \quad .$$

It is clear that the proper factor is  $\mu_n = \sqrt{\frac{2}{\pi}}$  for all  $n$ . Thus,  $\sqrt{\frac{2}{\pi}} \sin nx$  is a "normed" eigen-function. In general, if  $v_n$  is an eigen-function corresponding to  $\lambda_n$ , then so also is  $\mu_n v_n$ . We choose the constant  $\mu_n$  such that

$$\mu_n^2 \int_B \rho v_n^2 d\xi \quad \text{---} = 1 \quad ,$$

namely,

$$\mu_n = \sqrt{\frac{1}{c_n}} \quad .$$

If, now, we denote the functions  $\sqrt{1/c_n} v_n$  again by  $v_n$ , we then say that the eigen-functions  $v_n$  are "normed";

$$(7) \quad \int_B \rho v_n^2 d\xi \quad \text{---} = 1 \quad .$$

In effect, from among all possible eigen-functions (differing by a constant factor) corresponding to an eigen-value  $\lambda_n$  we denote by  $v_n$  that one which satisfies (7).

Returning to the question of orthogonality, we forego a general proof and merely indicate the nature of the proof in a few special cases. For the non-homogeneous string with fixed endpoints (treated in section 4 of part V), the equation of motion is

$$(8) \quad v'' + \lambda \rho v = 0 \quad .$$



Thus, we have

$$\lambda_n \int_0^{\ell} \rho v_n v_m d\xi = - \int_0^{\ell} v_m v_n'' d\xi$$

and

$$\lambda_m \int_0^{\ell} \rho v_n v_m d\xi = - \int_0^{\ell} v_n v_m'' d\xi ,$$

whence

$$\begin{aligned} (\lambda_n - \lambda_m) \int_0^{\ell} \rho v_n v_m d\xi &= \int_0^{\ell} (v_n v_m' - v_m v_n') d\xi \\ &= \int_0^{\ell} (v_n v_m' - v_m v_n') d\xi \\ &= (v_n v_m' - v_m v_n') \Big|_0^{\ell} = 0 \end{aligned}$$

by virtue of the boundary conditions  $v(0) = v(\ell) = 0$ . On the assumption that the eigen-values are single-valued it follows that  $\lambda_n \neq \lambda_m$  and therefore

$$\int_0^{\ell} \rho v_n v_m d\xi = 0 \quad \text{for } n \neq m .$$

Similarly, for the membrane with boundary fixed along a curve in a plane, the eigen-value equation is

$$\nabla^2 v + \lambda v = 0 ,$$

and this yields

$$(\lambda_n - \lambda_m) \iint_B v_n v_m d\xi d\eta = \iint_B (v_n \nabla^2 v_m - v_m \nabla^2 v_n) d\xi d\eta .$$

By Green's formula, the right member is equal to

$$\int_C (v_n \frac{\partial v_m}{\partial n} - v_m \frac{\partial v_n}{\partial m}) ds$$



which is equal to zero because of the boundary conditions. The orthogonality of the eigen-functions follows.

Exercises: 1. Prove the orthogonality property for the eigen-functions of the two examples of the string of section 5, part V.

2. Do the same for the rotating string of the exercise on page 125.

3. Do the same for the membrane with the boundary condition

$$\frac{\partial u}{\partial n} + \sigma u = 0 \quad .$$

4. Do the same for the clamped plate:

$$\nabla^2 \nabla^2 u - u_{tt} = 0 \quad \text{in } B, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } C \quad ;$$

also, for the clamped cantilever:

$$u''' - u_{tt} = 0 \quad , \quad u(0) = u(1) = u'(0) = u'(1) = 0 \quad .$$

Another important property possessed by the set of eigen-functions is that of completeness. For the homogeneous string with fixed endpoints the theory of Fourier series demonstrated that any reasonable motion corresponding to a given initial state could be represented by a superposition of the natural vibrations, i.e., vibrations of the form  $\sin nk'x \cdot e^{inkt}$  ( $k'$  and  $k$  constants). In other cases, where the eigen-functions were not simple harmonic vibrations, we mentioned that an extension of the concept of Fourier series would enable us to represent a motion by a superposition or development of the eigen-functions whether they were sine or cosine functions or not. This extension or generalization of the idea of Fourier series leads to a rather profound property of the eigen-functions, the proof of which is beyond the scope of this course. Nevertheless, we state this property and briefly indicate its significance.

The eigen-functions of any given eigen-value problem, such as those already treated by us, form a complete system of functions.





Roughly, completeness means that any reasonable\* function  $\varphi(x, \dots)$ , defined in a domain B and satisfying the prescribed boundary conditions of the problem\*\*, may be approximated to any specified degree of accuracy by a linear combination

$$\gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_n v_n$$

of the eigen-functions  $v_1, v_2, \dots, v_n, \dots$  of the given problem,  $\gamma_1, \gamma_2, \dots, \gamma_n$  being constants. To be precise we should state that this approximation is not measured linearly but by the mean square deviation

$$(9) \quad \int_B \rho \left( \varphi - \sum_{\nu=1}^n \gamma_\nu v_\nu \right)^2 d\xi \quad \text{---} .$$

As a matter of fact, the measurement of the accuracy of the approximation in this manner leads immediately to a Fourier-like expansion of  $\varphi(x, \dots)$  in terms of the eigen-functions. That is,  $\varphi(x, \dots)$  may be represented by a superposition

$$(10) \quad \sum_{\nu=1}^{\infty} a_\nu v_\nu(x, \dots)$$

of the eigen-functions  $v_\nu(x, \dots)$ , where the coefficients  $a_\nu$  are given, as in the case of the ordinary Fourier series, by

$$(11) \quad a_\nu = \int_B \rho \varphi v_\nu d\xi \quad \text{---} .$$

This fact appears immediately when we consider the problem of determining the values of the  $\gamma_\nu$  for which the mean square deviation (9) is a minimum. Squaring out the integrand of (9) and using the orthogonality relations (6) and (7), we have

\* For example, a piece-wise continuous function.

\*\* As we mentioned in the case of Fourier series, this restriction may be removed.



$$\int_{\text{---}} \int_B \rho \left( \varphi - \sum_{\nu=1}^n x_{\nu} v_{\nu} \right)^2 d\xi \text{ ---}$$

$$= \int_{\text{---}} \int_B \rho \varphi^2 d\xi \text{ ---} + \sum_{\nu=1}^n x_{\nu}^2 - 2 \sum_{\nu=1}^n x_{\nu} \int_{\text{---}} \int_B \rho \varphi v_{\nu} d\xi \text{ ---}.$$

By (11) this becomes

$$(12) \quad \int_{\text{---}} \int_B \rho \varphi^2 d\xi \text{ ---} + \sum_{\nu=1}^n (x_{\nu} - a_{\nu})^2 - \sum_{\nu=1}^{\infty} a_{\nu}^2.$$

Since the middle term of the last expression is always non-negative, (9) will clearly be a minimum when this term vanishes, i.e., when  $x_{\nu} = a_{\nu}$ . In other words, the mean square deviation (9) is a minimum when the development coefficients are given by (11).

This then leads to the expansion (10) which "represents"  $\varphi(x, \dots)$  in the sense that, of all the expansions in terms of the eigen-functions, (10) is the one for which each partial sum gives the best mean square approximation of the function  $\varphi(x, \dots)$ . It should be made clear, however, that this does not mean that  $\varphi(x, \dots)$  is equal to the sum (10). Nevertheless, it may be shown that the approximation may be made to any desired degree of accuracy simply by taking a sufficiently large number of eigen-functions in the expansion. Mathematically, this is expressed by

$$(13) \quad \lim_{n \rightarrow \infty} \int_{\text{---}} \int_B \rho \left( \varphi - \sum_{\nu=1}^n a_{\nu} v_{\nu} \right)^2 d\xi \text{ ---} = 0.$$

Referring to (12), in which we set  $x_{\nu} = a_{\nu}$ , we may write (13) in the following form which is the usual expression for the completeness of a set of functions, namely,

$$(14) \quad \int_{\text{---}} \int_B \rho \varphi^2 d\xi \text{ ---} = \lim_{n \rightarrow \infty} \sum_{\nu=1}^n a_{\nu}^2 = \sum_{\nu=1}^{\infty} a_{\nu}^2.$$

Even the completeness of the set of eigen-functions does not imply the equality of  $\varphi(x, \dots)$  and the expression (10). For, as is well-known from the calculus, this can occur only



when, in (13), the limiting process and the integration may be interchanged. The conditions under which the equality is possible will not be considered here. Suffice it to say that further study leads to developability conditions analogous to those for Fourier series.

Exercise: By means of the completeness relation (14) prove that  $\varphi(x, \dots)$  is uniquely determined by its development coefficients. Do this by showing that a function  $\Psi(x, \dots)$  all of whose development coefficients are identical with those of  $\varphi(x, \dots)$  satisfies the relation  $\Psi = \varphi$  in B.

## 2. Equivalence of the Impulse Method and the Expansion Method for External Forces

In sections 2 and 3 of part V, two methods were developed for finding a particular solution of the non-homogeneous equation

$$(15) \quad L[u] - u_{tt} = f(x, \dots; t) \quad *$$

relating to vibrations under an external force. Although the two methods - the expansion method and the method of impulses - appeared in essentially different ways, we mentioned that intrinsically they were equivalent and, furthermore, we even verified this equivalence in a special case. We now proceed to demonstrate this equivalence for the case of the general problem of motion given by (15).

The expansion method, it will be remembered, reduced the problem of finding a particular solution of (15) to the problem of solving a sequence of ordinary differential equations. Let us suppose that we have solved the problem of free motion associated with (15) and know, therefore, the eigen-values  $\lambda_1, \lambda_2, \dots$  and the eigen-functions  $v_1, v_2, \dots$  of

$$(16) \quad L[v] + \lambda v = 0 \quad .$$

Then, by the developability property, we may expand  $f(x, \dots; t)$  in terms of the eigen-functions,

---

\* We let  $\rho = 1$  in this discussion for the sake of simplicity.



$$(17) \quad f(x, \dots; t) = \sum_{n=1}^{\infty} A_n(t) v_n(x, \dots)$$

where

$$(18) \quad A_n(t) = \int_B f(\xi, \dots; t) v_n(\xi, \dots) d\xi \quad .$$

We may assume the required particular solution of (15) to be of the form

$$(19) \quad u(x, \dots; t) = \sum_{n=1}^{\infty} U_n(t) v_n(x, \dots) \quad ,$$

where the  $U_n(t)$  are to be determined by substitution into the differential equation (15). Thus, we have

$$L[v_n] U_n - v_n \ddot{U}_n = A_n(t) v_n$$

or, since  $L[v_n] + \lambda_n v_n = 0$  ,

$$(20) \quad \ddot{U}_n + \lambda_n U_n = -A_n(t) \quad , \quad (n = 1, 2, \dots) \quad .$$

This is a system of ordinary differential equations which we can solve for the  $U_n(t)$ . Since the particular solution is to have zero initial conditions we must require that  $U_n(0) = \dot{U}_n(0) = 0$ . It is readily seen that the required solutions of (20) are

$$(21) \quad U_n(t) = \frac{1}{\omega_n} \int_0^t \sin \omega_n(t - \tau) A_n(\tau) d\tau \quad ,$$

where  $\omega_n = \sqrt{\lambda_n}$  are the natural frequencies. Replacing the  $U_n$  in (19) by the values given by (21) we arrive at the desired particular solution, since (19) also satisfies the required homogeneous boundary condition by virtue of the fact that the  $v_n$  have this property.

The impulse method, on the other hand, reduces the finding of the required particular solution of (15) to the problem of determining a solution  $u = H(x, \dots; t; \tau)$  of the corresponding homogeneous equation, a solution which satisfies the required boundary conditions and the initial conditions





$$(22) \quad \begin{cases} u(x, \dots; \tau) = 0 \\ u_t(x, \dots; \tau) = f(x, \dots; \tau) \end{cases} .$$

According to our general theory, the solution of this problem may be given by a superposition of the natural modes of the free motion, i.e.,

$$H(x, \dots; t; \tau) = \sum_{n=1}^{\infty} (a_n \cos \omega_n(t - \tau) + b_n \sin \omega_n(t - \tau)) v_n(x, \dots)$$

where we need only determine the  $a_n$  and  $b_n$  so that (22) is satisfied. We find immediately that  $a_n = 0$  and

$$(23) \quad f(x, \dots; \tau) = \sum_{n=1}^{\infty} \omega_n b_n v_n(x, \dots) ,$$

whence

$$b_n = \frac{1}{\omega_n} \int_B \dots \int f(\xi, \dots; \tau) v_n(\xi, \dots) d\xi \dots .$$

The function  $H(x, \dots; t; \tau)$  is thus determined and, according to section 3 of part V, the required solution of (15) is

$$(24) \quad u(x, \dots; t) = \int_0^t H(x, \dots; t; \tau) d\tau .$$

The equivalence of (24) and the solution obtained by the expansion method is immediate if we compare (23) with (17) and observe that, because of the uniqueness of the development of  $f(x, \dots; t)$

$$(25) \quad A_n(\tau) = \omega_n b_n ,$$

whence,

$$(26) \quad H(x, \dots; t; \tau) = \sum_{n=1}^{\infty} \frac{1}{\omega_n} A_n(\tau) \sin \omega_n(t - \tau) v_n(x, \dots) ,$$

and



$$(27) \quad u(x, \dots; t) = \sum_{n=1}^{\infty} \frac{1}{\omega_n} v_n(x, \dots) \int_0^t \sin \omega_n(t - \tau) A_n(\tau) d\tau .$$

A glance at (21) and (19) with (27) reveals immediately the identity of the results of the two methods.

### 3. Expansion of Green's Function in Terms of the Eigen-functions

When considering problems of equilibrium we introduced the theoretically useful concept of Green's function,

$$K(x, \dots; \xi, \dots) .$$

We saw that  $K$  is a function  $K(P; Q)$  of two points  $P: (x, \dots)$  and  $Q: (\xi, \dots)$ . If  $P \neq Q$ , then  $K$  satisfies the equation

$$L[K] = 0 ,$$

while for  $P = Q$ ,  $K$  or one of its derivatives possesses a singularity. Furthermore,  $K$  attains the prescribed homogeneous boundary values imposed on the forced system

$$(29) \quad L[u] = f(x, \dots) .$$

The solution of the equilibrium problem relating to (29) was found to be

$$(30) \quad u(x, \dots) = \int_B \int \dots \int K(x, \dots; \xi, \dots) f(\xi, \dots) d\xi \dots .$$

We now proceed to obtain a very simple expansion of  $K(x, \dots; \xi, \dots)$  in terms of eigen-functions. Suppose, first of all, that the eigen-values and eigen-functions of the vibrating system

$$L[u] - u_{tt} = 0$$

corresponding to (29) have been found. We may then develop the functions  $f$  and  $u$  in terms of the eigen-functions:



$$f(x, \dots) = \sum_{n=1}^{\infty} A_n v_n(x, \dots) \quad ,$$

$$u(x, \dots) = \sum_{n=1}^{\infty} a_n v_n(x, \dots) \quad .$$

Substituting these expressions into (29) we obtain

$$\sum_{n=1}^{\infty} a_n L[v_n] = \sum_{n=1}^{\infty} A_n v_n \quad .$$

Since  $L[v_n] + \lambda_n v_n = 0$ , we then have

$$\sum_{n=1}^{\infty} a_n \lambda_n v_n = - \sum_{n=1}^{\infty} A_n v_n \quad ,$$

whence,

$$(31) \quad a_n = - \frac{A_n}{\lambda_n} = - \frac{1}{\lambda_n} \int_B \int f(\xi, \dots) v_n(\xi, \dots) d\xi \quad \dots \quad .$$

Therefore,

$$u(x, \dots) = - \sum_{n=1}^{\infty} \frac{v_n(x, \dots)}{\lambda_n} \int_B \int f(\xi, \dots) v_n(\xi, \dots) d\xi \quad \dots \quad .$$

Interchanging integration and summation and realizing that  $v_n(x, \dots)$  is independent of the integration variables  $\xi, \dots$ , we find

$$(32) \quad u(x, \dots) = - \int_B \int f(\xi, \dots) \sum_{n=1}^{\infty} \frac{v_n(x, \dots) v_n(\xi, \dots)}{\lambda_n} d\xi \quad \dots \quad .$$

Referring to (30), we conclude that

$$(33) \quad K(x, \dots; \xi, \dots) = - \sum_{n=1}^{\infty} \frac{v_n(x, \dots) v_n(\xi, \dots)}{\lambda_n} \quad .$$

This is the desired expansion.

The question of convergence cannot be discussed here. We are satisfied merely to find a formal relation between Green's function and the eigen-functions. We may, however, say that at least one fact is certain, namely that the mean square



sums of (33) can be made as small as we wish provided only that we take a sufficiently large number of terms in the partial sums.

Exercise: Check the general formula (33) for the case of a homogeneous string with fixed ends, by means of a well-known Fourier series.

#### 4. Asymptotic Behavior of the Eigen-values

We have seen that the natural frequencies  $\omega_n$  of a homogeneous string with fixed ends are proportional to the positive integers. In particular, if the string is of length  $\pi$  and if  $m = S = 1$ , then

$$(34) \quad \omega_n = n$$

and

$$(34') \quad \lambda_n = \omega_n^2 = n^2.$$

On the other hand, if the string is non-homogeneous or is not fixed at its endpoints, then the following is true: The eigen-values are at least asymptotically proportional to  $n^2$ . That is,

$$(35) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n}{n^2} = \beta$$

where  $\beta$  is a positive constant. Thus, for example, the string of section 5 of part V, in which one end is fixed and the other free to move along a vertical line, has the natural frequencies

$$\omega_n = \frac{2n - 1}{2}.$$

Hence,

$$\frac{\lambda_n}{n^2} = \frac{(2n - 1)^2}{4n^2} \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty.$$

This asymptotic phenomenon may be expressed in terms of the theory of sound, thus: A non-homogeneous string or a string with only one fixed endpoint will have overtones which are not exactly harmonic, but, as one proceeds higher and higher in the





scale of overtones, all the more will these overtones tend to become harmonic.

We know that the musical quality of a musical instrument depends upon the mixture of the intensities of the fundamental tone and the overtone and also upon the character of the overtones relative to the fundamental tone. Thus, one-dimensional instruments, such as string and wind instruments, are characterized by approximately harmonic overtones. On the other hand, two-dimensional instruments, such as drums, cymbals, church bells, etc., present a very different character of tone quality, because in these cases the overtones do not tend to become harmonic, but, rather, become much denser than those of one-dimensional instruments.

An excellent insight into this phenomenon is furnished by the study of the square membrane. For the sake of simplicity, we assume that the square is given by  $0 \leq y \leq \pi$ , and, further, that the mass density  $m$  and the tension  $S$  are both unity. The eigen-values for the membrane, with boundary fixed in the plane along the sides of the square, are, as we have already seen,

$$(36) \quad \lambda^{(i)} = n^2 + m^2, \quad (i; n, m = 1, 2, 3, \dots).$$

Let  $A(\lambda)$  denote the number of eigen-values which are numerically less than or equal to a given eigen-value  $\lambda$ . Then  $A(\lambda)$  is the number of positive integral roots of the inequality

$$(37) \quad n^2 + m^2 \leq \lambda.$$

If, in the  $\xi, \eta$ -plane, we draw the circle

$$\xi^2 + \eta^2 = \lambda,$$

then  $A(\lambda)$  is exactly the number of lattice points (with integral coordinates) which lie inside or on the circle in the first quadrant.

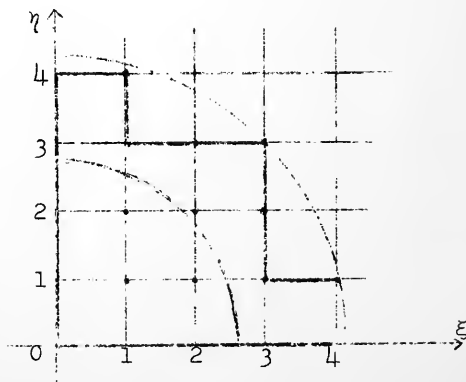


Fig. 1



The lattice points lying on the  $\xi$  or  $\eta$  axes are excluded since both  $m$  and  $n$  must be positive. We construct the rectilinear figure made up of all the squares of the quadratic mesh which possess an admissible lattice point in the upper right-hand corner. (See figure 1). Since this polygon is certainly included within the quarter circle and since its area is obviously  $A(\lambda)$ , we have

$$(38) \quad A(\lambda) < \frac{\pi\lambda}{4}.$$

In order to obtain another approximation for  $A(\lambda)$  we draw another circle

$$\xi^2 + \eta^2 = r^2$$

a quarter of which lies entirely within the polygon. Clearly, for sufficiently large  $\lambda$ , we need only take  $r = \sqrt{\lambda} - 2$  to satisfy this condition. We then have

$$\frac{\pi}{4} (\sqrt{\lambda} - 2)^2 < A(\lambda) < \frac{\pi\lambda}{4}$$

or

$$\frac{\pi}{4} \left(1 - \frac{4}{\sqrt{\lambda}} + \frac{4}{\lambda}\right) < \frac{A(\lambda)}{\lambda} < \frac{\pi}{4},$$

so that

$$(39) \quad \lim_{\lambda \rightarrow \infty} \frac{A(\lambda)}{\lambda} = \frac{\pi}{4}.$$

In other words, the number of eigen-values less than or equal to  $\lambda$  is asymptotically proportional to  $\lambda$ ,

$$(40) \quad A(\lambda) \sim \frac{\pi}{4} \lambda.$$

Another form of (40), which illustrates more closely the density of distribution of the eigen-values, is obtained as follows. Let  $\lambda = \lambda^{(k)}$ , the  $k^{\text{th}}$  eigen-value. Then  $A(\lambda^{(k)}) = k$ , and (40) becomes

$$(40') \quad \lambda^{(k)} \sim \frac{4}{\pi} k$$

or

$$(39') \quad \lim_{k \rightarrow \infty} \frac{\lambda^{(k)}}{k} = \frac{4}{\pi}.$$



If we compare (39') with relation (35) for the string, we observe that, in the case of the membrane, the eigen-values tend to infinity much slower, i.e., as  $1, 2, 3, \dots$ , than do those for the string which tend to infinity like the sequence  $1^2, 2^2, 3^2, \dots$ . Thus, the natural frequencies of the membrane are denser than those of a string with the same fundamental tone.

It turns out that this phenomenon of the relative densities of distribution of the natural frequencies of different types of instruments is even more marked in the case of a vibrating plate, a case which we shall not discuss here.

A very interesting theoretical result, which we merely mention in passing, is that the asymptotic law is essentially the same for membranes of all shapes. That is

$$(41) \quad A(\lambda) \sim \frac{\Delta B}{4\pi} \lambda$$

where  $\Delta B$  is the area of the general domain  $B$ . From this it follows that the asymptotic distribution of the natural frequencies does not depend upon the shape of the membrane or upon the imposed boundary conditions, but merely upon the area of the domain over which the membrane is stretched.

These results, which can be extended to many other types of differential equations, have proved to be of great importance in the quantum theory of specific heat and radiation. The corresponding proofs for more general equations and domains, by the maximum-minimum property of the eigen-values, will be discussed in the next chapter.

Exercises: 1. Prove the asymptotic law (41) for the rectangular membrane.

2. Show that  $\omega_n = n$ , where the  $\omega_n$  are the natural frequencies of the string discussed in the second example of section 5 of part V.



# CHAPTER THREE

## VARIATIONAL METHODS\*

### Part I. Introduction

In recent years the application of variational methods to problems of engineering has proved of great value. In this chapter we shall discuss these and similar devices and, in particular, the Rayleigh-Ritz method. However, before proceeding to a systematic discussion, we shall give a brief summary of the interesting history of the Calculus of Variations.

In the year 1696, after the theory of ordinary maxima and minima for functions of a finite number of variables had already played a very decisive role in the early stages of the differential and integral calculus, the general attention of mathematicians of that era was called to a new type of minimum problem through a public challenge issued by the Swiss mathematician John Bernoulli. He proposed for solution the following problem of the brachistochrone.

Among all paths joining points A and B (see figure 1), find that path along which a mass particle, subject only to the influence of gravity, will travel from A to B in the shortest possible time.

The novelty of this problem consists in the fact that the quantity to be minimized depends not on a finite number of independent variables, but, rather, on the behavior of a whole curve or function, which, naturally, requires the knowledge of "infinitely-much data". It is an easy matter to formulate the mathematical expression to be minimized in the brachistochrone problem. Let  $y = y(x)$  be a

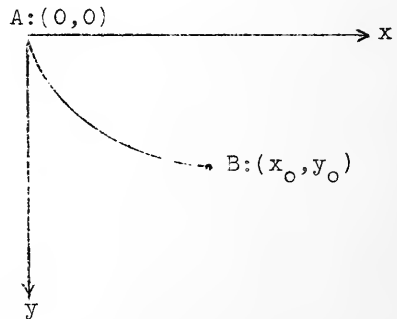


Fig. 1

---

\* This chapter is only a brief survey and preliminary outline of these important methods. A more detailed presentation is planned for a later course.





curve joining  $A(0,0)$  and  $B(x_0, y_0)$ . The velocity of the particle along this curve is given by  $\sqrt{2gy}$  and the total time to travel along this path from A to B is

$$(1) \quad T = \int_0^{x_0} \sqrt{\frac{1 + y'^2}{2gy}} dx.$$

The time  $T$  is, thus, "a function of the curve  $y = y(x)$ ". To solve the brachistochrone problem we must determine that curve  $y = y(x)$  for which the integral expression (1) has the smallest possible value, under the added conditions that  $y(0) = 0$  and  $y(x_0) = y_0$ .

The classical isoperimetric problem of finding that closed curve of given length which includes the largest possible area is a problem of essentially the same type. A simple consideration shows that such a curve must necessarily be convex and may be assumed to be symmetric about the  $x$ -axis, so that the original problem reduces to the problem of minimizing the area

$$(2) \quad A = 2 \int_0^{x_0} y(x) dx,$$

under the conditions

$$(3) \quad L = 2 \int_0^{x_0} \sqrt{1 + y'^2} dx$$

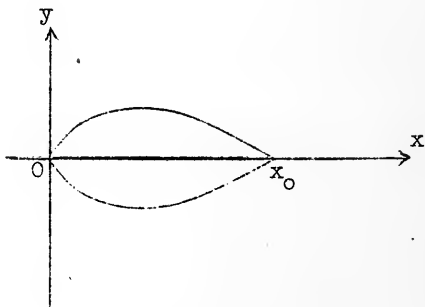


Fig. 2

and  $y(0) = y(x_0) = 0$ . The quantity  $L$  is the prescribed length of the curve.

Soon after the brachistochrone problem had been formulated and solved, mathematicians realized that many other problems fell into the same category. We formulate some typical problems.

a) Of all the curves of given length  $L$  suspended between two points A and B, find that one whose center of gravity is the lowest.



We may interpret this problem physically by supposing the curve to be a homogeneous chain of length  $L$ . Fulfilling the requirement that the center of gravity be as low as possible is equivalent to finding the position of stable equilibrium of the chain. Analytically, this amounts to finding a curve  $y = y(x)$  such that

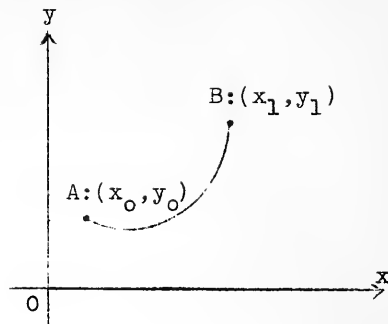


Fig. 3

$$(4) \quad \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx$$

is as small as possible, while the curve has the prescribed length

$$L = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx$$

and passes through the points A and B.

b) Another problem which attracted attention during the early days of the development of variational theory was what we might today call the problem of "streamlining". It is required to find what the shape of a projectile should be in order that a fluid offer the least possible resistance to its motion.

c) About this time, Fermat formulated the principle that light travels from one point to another in such a way that the time of transmission is a minimum with respect to the times required along all other "virtual" paths joining the two given points. From this principle it follows immediately that in a homogeneous medium light travels in a straight line. Likewise, the law of reflection is a simple geometric consequence of Fermat's principle. All that need be shown is that if two given points A and B on the same side of a curve C are joined by a path touching C, then the shortest such path is the broken



line APB, where the angles  $\alpha_1$  and  $\alpha_2$  between C and AP and C and BP, respectively, are equal. Similarly, one may derive the well-known law of refraction, which states that if C divides the plane into two media with corresponding light velocities  $c_1$  and  $c_2$ , then the light traveling from A to D is refracted in such a manner that  $\sin \beta_1 / \sin \beta_2 = c_1 / c_2$ .

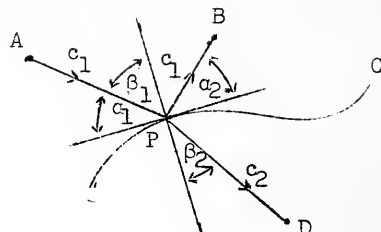


Fig. 4

By means of Fermat's principle, we may formulate the general problem of finding the path of light in a planar non-homogeneous medium in which the velocity of light  $\varphi(x,y)$  varies continuously. Since the path of light joining two points  $A:(x_0, y_0)$  and  $B:(x_1, y_1)$  is characterized by the fact that the time  $T$  of transmission is a minimum, we must determine that curve  $y = y(x)$  joining  $A$  and  $B$  for which

$$(5) \quad T = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\varphi(x, y)} dx$$

is a minimum.

At first, the early pioneers in this new field, such as the Bernoulli brothers and Euler, handled problems of the above type by various special and ingenious methods and devices, applying, in most cases, only to the given particular problem. Although they undoubtedly recognized that these maxima and minima problems possessed a certain novelty not contained in problems of ordinary maxima and minima, they do not evolve a general theory. It remained for Lagrange, a short time later, to formulate general principles of attack for these problems. His idea was to reduce the problem of minimizing or maximizing the given integral expression containing the unknown function to the task of solving a boundary value problem involving two



differential equations. The manner in which this reduction is accomplished will be taken up in succeeding sections of this chapter.

The significance of variational methods in mechanical and physical theory was first sensed in a rather vague and mystical manner in the 18-th century. One wanted to express all natural laws to agree with the premise that "God does everything in the best possible way". The new variational theory thus offered a very convenient tool to express natural laws as the maxima or the minima of "certain classes of virtual phenomena" according to the case in mind. However, the mysticism which shrouded such an "optimum principle" rather soon gave way to a new and deeper insight into the scientific study of these phenomena. It was Euler who divested the theory of its mysticism by formulating the laws of mechanics by means of a so-called "variational principle" instead of by means of an "optimum principle". The Euler principles gained more and more prominence in the 19-th century after Hamilton had introduced them in the theory of optics. Stimulated by marked successes, scientists discovered that variational problems not only were related to large fields of mathematics but also were important for a deeper penetration into the physical laws of nature.

Rayleigh was the first to use, with success, the variational theory for numerical solutions to problems of vibrations. Independently, the Swiss mathematical-physicist Ritz attacked similar problems with great success. For example, until that time no satisfactory theoretical explanation of the strange nodal lines of vibrating plates had been given. Ritz, by the application of variational methods, actually succeeded in providing the desired explanation. Since then, the Rayleigh-Ritz method (which is the usual designation for the variational methods as applied by Rayleigh and Ritz) has assumed a significant place in engineering and physics. Much of the work of Professor Timoshenko at Stanford makes use of this method as does the work of other great contemporary engineers such as von Karman. As a matter of fact, it seems that even much more could be achieved if not only the Rayleigh-Ritz patterns, but also other possibilities which present themselves naturally would be applied by engineers.





## Part II. Mathematical Preparation

### 1. Homogeneous Quadratic "Functionals"

We shall solve problems of equilibrium and of motion by minimizing certain quantities or, more precisely, by making these quantities "stationary". The quantities to be considered will be given by expressions involving the reduced forms of the kinetic and potential energies of a physical system. Under the assumption of small deflections, these expressions will have the form of "homogeneous quadratic functionals". For example, we have seen that for a membrane with fixed boundary the reduced potential energy is given by

$$(1) \quad \bar{V} = \int_B \int (v_x^2 + v_y^2) \, dx \, dy \quad *$$

where  $v$  is a function only of  $x$  and  $y$ , and  $B$  is the domain of the  $x, y$ -plane over which the membrane is stretched. This expression for  $\bar{V}$ , which we shall denote by  $\bar{V}[v, v]$  is homogeneous and quadratic in the "independent variable  $v$ " in the sense that

$$\bar{V}[cv, cv] = c^2 \bar{V}[v, v]$$

if  $c$  is a constant. Furthermore, if in (1) we let  $v(x, y) = p(x, y) + q(x, y)$ , then

$$\bar{V}[p+q, p+q] = \bar{V}[p, p] + \bar{V}[q, q] + 2\bar{V}[p, q] \quad .$$

The quantity

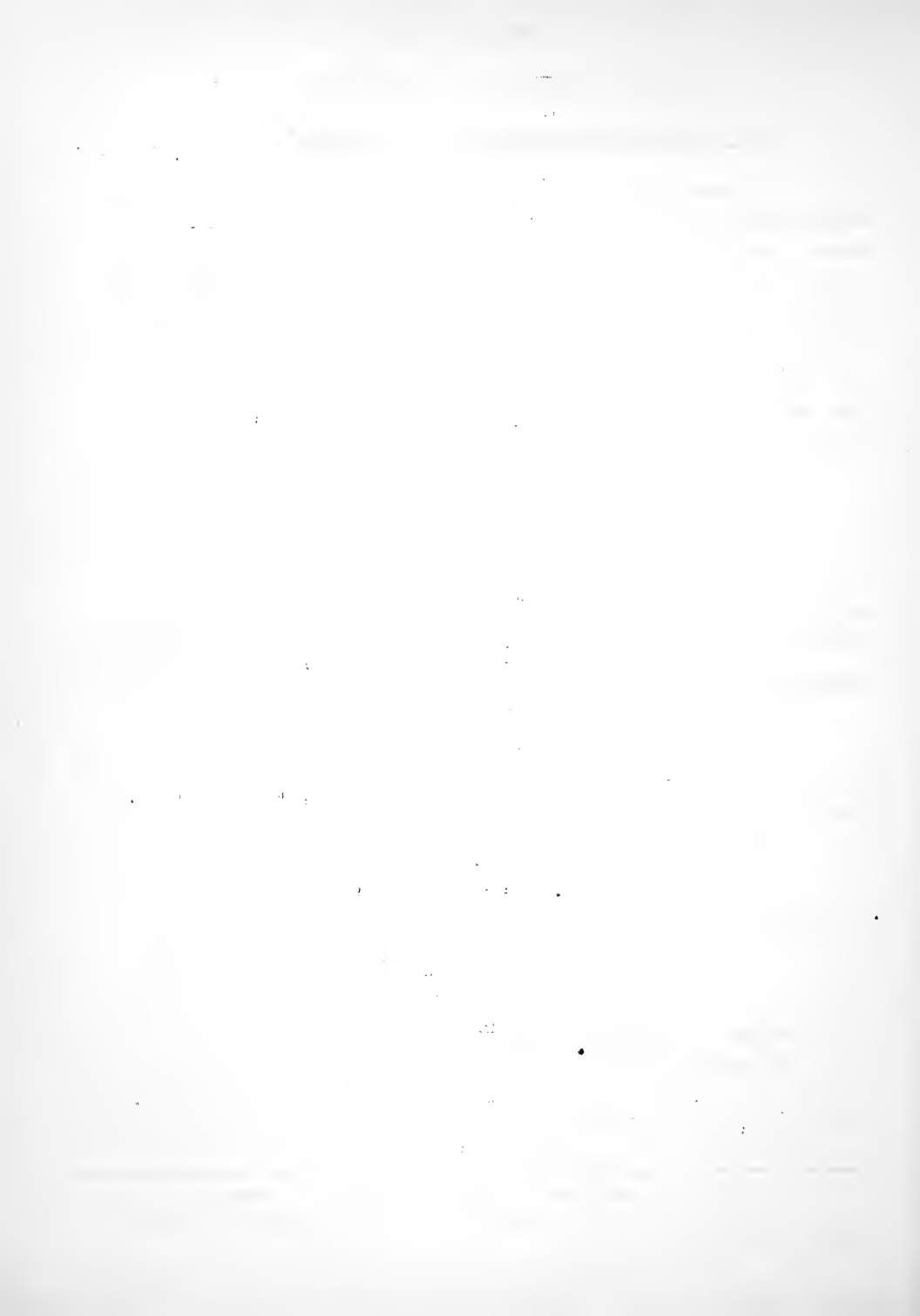
$$\bar{V}[p, q] = \int_B \int (p_x q_x + p_y q_y) \, dx \, dy$$

is the so-called mixed or polar form.

In general, we shall encounter, in what follows, various homogeneous quadratic functionals involving a variable function  $v(x, y)$ . For such expressions we will use the general notation

---

\* Throughout this chapter we shall assume, for simplicity, that  $S = m = 2$ . Thus, in (1), the factor of the integral is unity instead of  $S/2$ .



$Q[v, v]$ . In order to become more acquainted with expressions of this type and, also, for the sake of future reference, we list here the reduced potential energies for various continuous mechanical systems.

a) The homogeneous string with fixed ends:

$$(2) \quad Q[v, v] = \int_0^{\ell} v'(x)^2 dx \quad .$$

b) The homogeneous membrane with fixed boundary:

$$(3) \quad Q[v, v] = \iint_B (v_x^2 + v_y^2) dx dy \quad .$$

c) The homogeneous clamped cantilever: The potential energy of any sufficiently small element of the deflected cantilever is proportional to the square of its curvature. Thus,

$$(4) \quad Q[v, v] = \int_0^{\ell} v''(x)^2 dx \quad .$$

d) The homogeneous clamped plate: We state without proof that the potential energy is given by the integral of a certain quadratic form in the principle curvatures  $1/\rho_1$  and  $1/\rho_2$  of the deflected plate. Denoting this quadratic form by

$$\alpha \left( \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) + \beta \frac{1}{\rho_1 \rho_2}$$

we have, by our assumption of small deflections,

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{2} \nabla^2 u \quad , \quad \frac{1}{\rho_1 \rho_2} = u_{xx} u_{yy} - u_{xy}^2 \quad .$$

Thus, the reduced potential energy of the plate, within a constant factor, may be written in the form

$$(5) \quad Q[v, v] = \iint_B \left[ (\nabla^2 v)^2 - 2(1-\delta)(v_{xx} v_{yy} - v_{xy}^2) \right] dx dy$$



where  $\gamma$  depends on  $\alpha$  and  $\beta$  and is determined as a material factor of the plate. Incidentally, we may mention at this time a fact which will become evident later on, namely; instead of using the reduced energy (5) for the clamped plate, we may, for our purposes, consider the simpler expression

$$(6) \quad Q[v, v] = \int_B \int (\nabla^2 v)^2 dx \, dy \quad .$$

We now reconsider some of these cases, removing, however, the condition that the boundaries of the systems are fixed. Rather, we assume that a prescribed force is applied along the boundary in each case, for example, an elastic force.

e) The string: Let  $h_0$  and  $h_1$  be the constant elastic forces at  $x = 0$  and  $x = \ell$ , respectively. The expression for the reduced potential energy furnished by these forces is

$$(7) \quad E[v, v] = h_0 v(0)^2 + h_1 v(\ell)^2 \quad ,$$

and the total reduced potential energy for the system becomes

$$(8) \quad \bar{V}[v, v] = Q[v, v] + E[v, v]$$

where  $Q$  is given by (2).

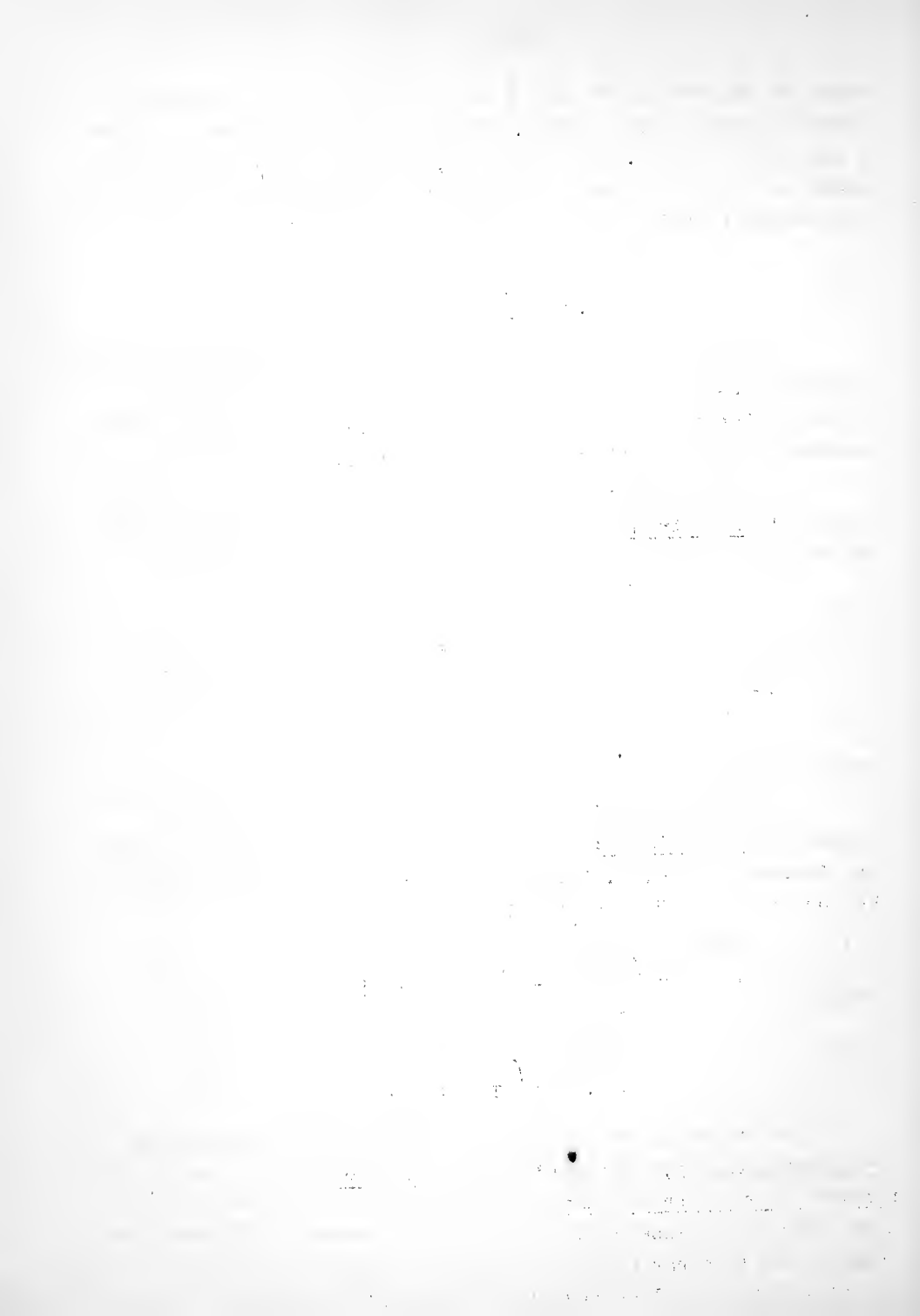
f) The membrane: Here we may denote the prescribed elastic force by  $\sigma(s)$ ,  $S$  being the arc-length measured along the boundary  $C$  of  $B$ . The total reduced energy for the membrane is

$$(9) \quad \bar{V}[v, v] = \int_B \int (v_x^2 + v_y^2) dx \, dy + E[v, v]$$

where

$$(10) \quad E[v, v] = \int_C \sigma v^2 ds \quad .$$

g) Finally, we consider a homogeneous clamped plate reinforced by a bar or cantilever along the line  $y = 0$ . We will assume that this line cuts the boundary  $C$  in only two points. The potential energy of this mechanical system is obviously the sum of the potential energy of a clamped plate and the energy



contributed by the cantilever reinforcement. The reduced form for the energy of the latter is

$$(11) \quad E[v, v] = \int_{x_0}^{x_1} (v_{xx}(x, 0))^2 dx \quad .$$

Hence, the total reduced energy for the system is

$$\bar{V}[v, v] = Q[v, v] + E[v, v] \quad ,$$

where  $Q[v, v]$  is given by (5).

All the energy expressions above enjoy one and the same formal property, which, incidentally, is the mathematical definition of a quadratic functional, namely,

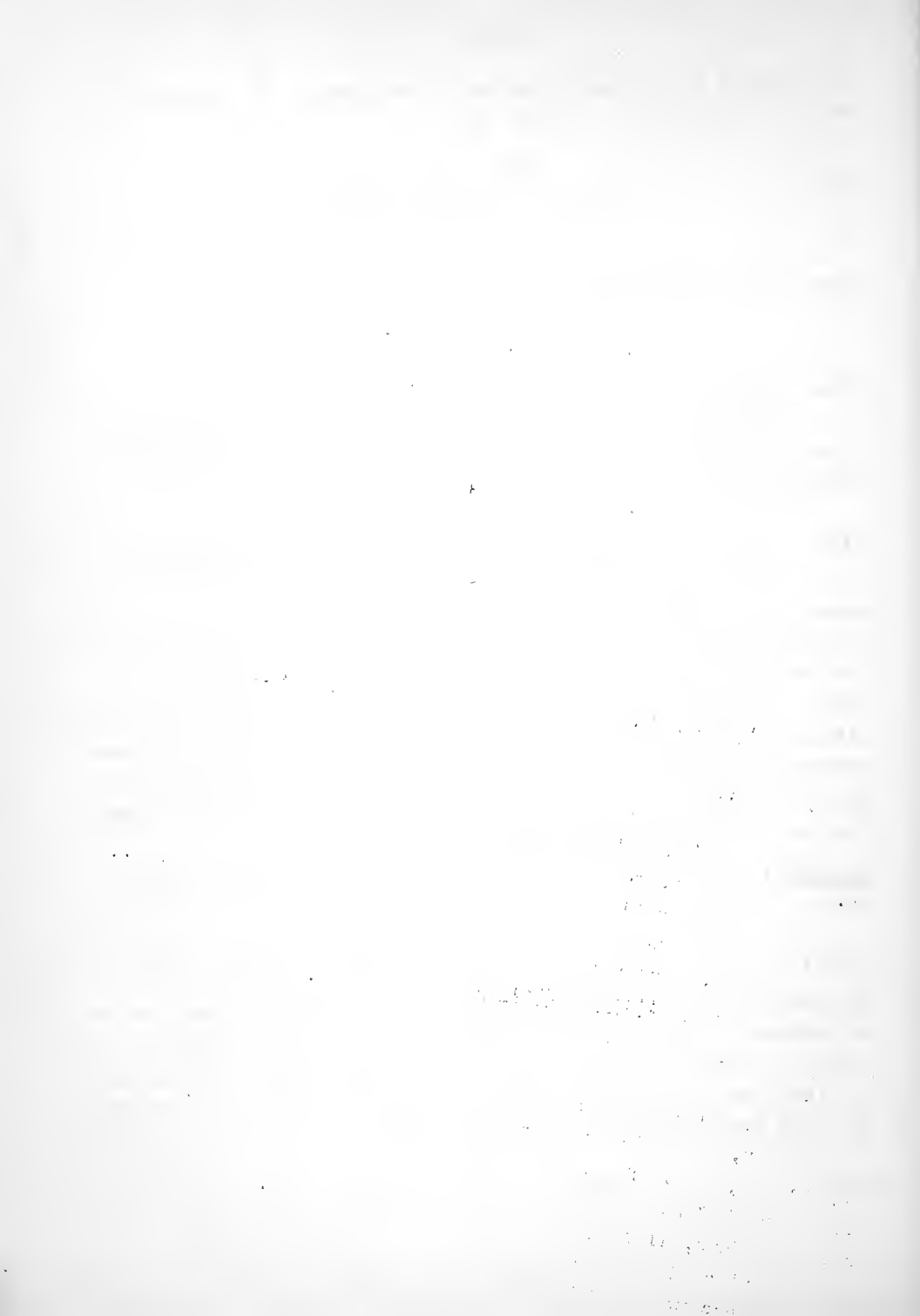
$$(12) \quad Q[\lambda v + \mu w, \lambda v + \mu w] = \lambda^2 Q[v, v] + 2\lambda\mu Q[v, w] + \mu^2 Q[w, w]$$

where  $\lambda$  and  $\mu$  are two arbitrary constants and  $v$  and  $w$  are any two functions for which the expression  $Q$  may be constructed. The polar form  $Q[v, w]$  is symmetric in  $v$  and  $w$  and linear in each. In addition, we shall assume that the quadratic expression  $Q[v, v]$  is never negative for any "admissible" function  $v$ , i.e., for any  $v$  satisfying the conditions of the given problem in which  $Q$  occurs. Quadratic expressions possessing this property are said to be positive definite.

Exercise: Find the polar expressions for each of the forms above.

In addition to the quadratic expressions in the variable function  $v$ , we shall encounter linear expressions in  $v$ . Physically, these forms will usually represent the work done by an external force applied to some portion of the mechanical system. Thus, if  $f(x, y)$  is the density of a force applied to the interior of a membrane or plate, then the reduced form of the work accomplished by this force is

$$(13) \quad H[v, f] = \int_B \int v f \, dx \, dy \quad .$$





## 2. Technicalities of the Calculus of Variations

In the variational problems under consideration, the general procedure will be to minimize or, more precisely, to make "stationary", under given boundary conditions, certain quadratic functionals or combinations of the same with linear functionals. To say that a quadratic functional is "to be made stationary" means the following. We consider a family of functions  $v(x,y;\epsilon)$  involving a parameter  $\epsilon$ , each function of the family satisfying the same boundary conditions as the required function  $v(x,y) = v(x,y;0)$ . Such a family constitutes the set of functions "admissible" to consideration in the variational problem. If, in the quadratic functional  $Q[v,v]$ , we replace  $v(x,y)$  by  $v(x,y;\epsilon)$ , then  $Q$  becomes a function of the parameter  $\epsilon$ . We denote it by  $Q(\epsilon)$ . The functional  $Q[v,v]$  is then said to be stationary if  $Q(\epsilon)$  for  $\epsilon = 0$  is either a minimum or a maximum with respect to  $\epsilon$ , namely, if

$$\left. \frac{dQ}{d\epsilon} \right|_{\epsilon=0} = 0.$$

In particular, we restrict the class of admissible functions to those functions  $v(x,y;\epsilon)$  which are of the form  $v(x,y) + \epsilon h(x,y)$  where  $h(x,y)$  is arbitrary within the limits set by the requirement that  $v + \epsilon h$  must satisfy the prescribed boundary and continuity conditions\*. It is easily seen, by replacing  $v$  by  $v + \epsilon h$  in  $Q[v,v]$ , that, by virtue of (12), a necessary condition for the stationary character of  $Q$  is

$$\left. \frac{dQ}{d\epsilon} \right|_{\epsilon=0} = 2Q[v,h] = 0,$$

or

$$(14) \quad Q[v,h] = 0.$$

Likewise, for the linear form  $H[v,f]$ , the stationary condition becomes

---

\* This restriction on the class of admissible functions offers no loss of generality, as one may easily verify by setting  $h(x,y) = \frac{\partial}{\partial \epsilon} v(x,y;\epsilon)$  for  $\epsilon = 0$ .



$$(15) \quad H[h, f] = 0 \quad .$$

It is possible, and a rather simple matter, to reduce these conditions for the stationary character of the above functionals to a boundary value problem for differential equations. For example, (14) states that the polar form  $Q[v, h]$  should vanish for all permissible but otherwise arbitrary functions  $h(x, y)$ . It is from the arbitrary character of  $h$  that the desired conclusions can be drawn in any specific case. To illustrate this, suppose that  $v + \xi h$  is admissible provided that  $h$  vanishes on the boundary  $C$  of the domain  $B$ , in addition to satisfying the required continuity and differentiability conditions. In any specific case, the task is, then, to reduce the functional  $Q[v, h]$  to the form

$$(16) \quad \iint_B hL[v] \, dx \, dy$$

where  $L[v]$  is a linear differential expression in the required function  $v(x, y)$ . Then, if (16) is to vanish for all permissible functions  $h$ , it follows, from the continuity of  $L[v]$  as a function of  $x$  and  $y$ , and from the arbitrariness of  $h$ , that

$$(17) \quad L[v] = 0 \quad \text{in } B \quad .$$

The proof of this statement is well-known.

The only point that remains to be considered is the actual reduction of  $Q[v, h]$  to the form (16). This is ordinarily accomplished by some form of integration by parts. We will illustrate the reduction by a few examples involving mechanical systems with fixed boundaries. If an external force  $f(x)$  or  $f(x, y)$ , as the case may be, acts on the system, then the resulting equilibrium position will be given by that function  $v(x)$  or  $v(x, y)$  which renders the potential energy of the system stationary. The stable equilibrium position is characterized by the fact that the potential energy is not only stationary but is actually a minimum with respect to all other admissible positions of the system.



a) The String: A string with fixed ends is characterized by the boundary conditions  $v(0) = v(\ell) = 0$ . The set of admissible functions  $v(x) + \xi h(x)$  must likewise possess these boundary values. Hence,  $h(0) = h(\ell) = 0$ . The potential energy of the string under a given external force of density  $f(x)$  is

$$(18) \quad Q[v, v] + H[v, f] = \int_0^{\ell} (v'^2 + vf) dx$$

so that

$$(19) \quad Q[v, h] + H[h, f] = \int_0^{\ell} (v'h + hf) dx.$$

If  $v(x)$  is to be the required equilibrium position, we must have

$$Q[v, h] + H[h, f] = 0$$

for all permissible  $h$ . In order to reduce (19) to the form (16) we need only integrate  $Q[v, h]$  by parts. Remembering that  $h(0) = h(\ell) = 0$ , we find that

$$Q[v, h] = - \int_0^{\ell} v'' h \, dx,$$

whence,

$$Q[v, h] + H[h, f] = - \int_0^{\ell} h(v'' - f) dx = 0$$

for arbitrary  $h$ , and hence,

$$(20) \quad v''(x) = f(x)$$

with the additional condition that  $v(0) = v(\ell) = 0$ .

b) The Cantilever: The clamped cantilever has the potential energy

$$(21) \quad Q[v, v] + H[v, f] = \int_0^{\ell} (v''^2 + vf) dx,$$

where  $v$  satisfies the boundary conditions

$$(22) \quad v(0) = v'(0) = v(\ell) = v'(\ell) = 0.$$



Hence, the admissible functions  $v + \varepsilon h$  have the property that  $h$  also satisfies the condition (22). The stationary condition

$$(23) \quad Q[v, h] + H[h, f] = \int_0^l (h'' v'' + hf) dx = 0$$

can be reduced to the boundary value problem

$$(24) \quad v''''(x) + f(x) = 0$$

with the boundary conditions (22). This reduction follows the same pattern as that in the case of the string, except that two integrations by parts are to be performed on  $\int_0^l h'' v'' dx$ .

c) The Membrane: The potential energy of a membrane stretched over a domain  $B$  and fixed along the boundary  $C$  of  $B$  is

$$(25) \quad Q[v, v] + H[v, f] = \int_B \int (v_x^2 + v_y^2 + vf) dx dy \quad .$$

The stationary character of (25) with respect to the admissible class  $v + \varepsilon h$ , for which  $h = 0$  along  $C$ , is expressed by

$$(26) \quad \int_B \int (v_x h_x + v_y h_y) dx dy + \int_B \int hf dx dy = 0$$

for arbitrary  $h$  within the class. We seek to reduce (26) to the form

$$\int_B \int h(L[v] + f) dx dy = 0$$

from which we may conclude, as before, that

$$L[v] + f = 0 \quad .$$

This is accomplished by the use of Green's formula, which when applied to the first integral of (26) yields

$$\int_B \int (v_x h_x + v_y h_y) dx dy = - \int_B \int h \nabla^2 v dx dy + \int_C h \frac{\partial v}{\partial n} ds \quad ,$$





where  $n$  is the outward normal to  $C$  and  $s$  the arc-length along  $C$ . However, since  $h$  vanishes along  $C$ , the last integral is zero. Thus, (26) becomes

$$-\int_B \int (\nabla^2 v - f) dx dy = 0, \quad ,$$

from which we see that the equilibrium position  $v(x,y)$  satisfies the differential equation

$$(27) \quad \nabla^2 v - f = 0$$

with the boundary condition  $v = 0$  on  $C$ .

d) The Plate: The procedure for a clamped plate is similar to that for the membrane. The boundary conditions are given by

$$(28) \quad v = \frac{\partial v}{\partial n} = 0 \quad \text{on } C.$$

Using expression (6), we see that the potential energy of the plate under an external force of density  $f(x,y)$  is

$$Q[v,v] + H[v,f] = \int_B \int (\nabla^2 v)^2 dx dy + \int_B \int v f dx dy$$

for which the stationary condition is

$$(29) \quad \int_B \int \nabla^2 v \nabla^2 h dx dy + \int_B \int h f dx dy = 0, \quad ,$$

the function  $h$  satisfying the boundary conditions (28). To apply Green's formula\* to the first integral of (29) we let  $\nabla^2 v = w$  and obtain for (29)

---

\* This formula, although different from the Green's formula used in the case of the membrane, is an immediate consequence of the latter. It states that

$$\int_B \int (w \nabla^2 u - u \nabla^2 w) dx dy = \int_C (w \frac{\partial v}{\partial n} - v \frac{\partial w}{\partial n}) ds.$$



$$\int_B \int h (\nabla^2 \nabla^2 v + f) dx dy + \int_C \left( h \frac{\partial \nabla^2 v}{\partial n} - \nabla^2 v \frac{\partial h}{\partial n} \right) ds = 0 .$$

Because of the boundary conditions imposed on  $h$ , the second integral vanishes and we obtain

$$(30) \quad \nabla^2 \nabla^2 v + f(x, y) = 0$$

as the differential equation for the equilibrium position of the plate under the condition (28).

Remark: We could have used the more complicated expression (5) for the potential energy of the plate. If this were done, then  $Q[v, h]$  would have the form

$$\int_B \int (\nabla^2 v \nabla^2 h + hf) dx dy - (1 - \gamma) \int_B \int (v_{xx} h_{yy} + v_{yy} h_{xx} - 2v_{xy} h_{xy}) dx dy.$$

If, then, the second integral in this expression is transformed to the form (16) by successive integrations by parts, one finds that

$$(31) \quad \int_B \int (v_{xx} h_{yy} + v_{yy} h_{xx} - 2v_{xy} h_{xy}) dx dy = \int_B \int h L[v] dx dy + \int_C M \frac{\partial h}{\partial n} ds + \int_C P h ds$$

where  $L[v] \equiv 0$  and  $M$  and  $P$  are expressions in the second derivatives of  $v$ . Explicitly,

$$(32) \quad M = u_{xx} x_n^2 + 2u_{xy} x_n y_n + u_{yy} y_n^2 ,$$

$$(33) \quad P = \frac{\partial}{\partial s} [u_{xx} x_n x_s + u_{xy} (x_n y_s + x_s y_n) + u_{yy} y_n y_s] ,$$

where  $x_n, y_n$  are the direction cosines of the outward normal to  $C$  and  $x_s, y_s$  the direction cosines of the tangent to  $C$ . Since we are now concerned with a clamped plate, the two boundary integrals in (31) vanish. Thus, we see that the additional term



$$(1 - \delta) \int_B \int (v_{xx} v_{yy} - v_{xy}^2) dx dy$$

in the potential energy of the clamped plate has no effect on the corresponding boundary value problem for the equilibrium position. On the other hand, if the plate is not clamped but, let us say, has a free boundary, then this additional term leads to a very significant result, namely, it determines the natural boundary conditions for the free plate, as we shall see in the next section. Incidentally, an expression like  $v_{xx} v_{yy} - v_{xy}^2$ , which has no effect on the differential equation expressing the stationary character of a given integral, is said to be a divergence expression.

### 3. Natural Boundary Conditions

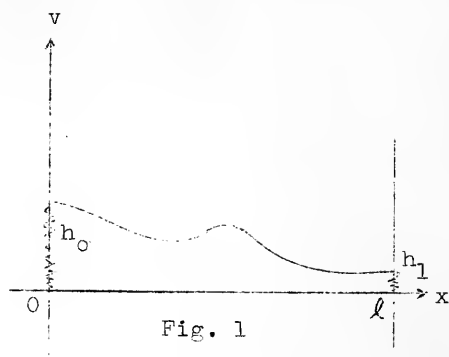
In the preceding section we considered cases in which precise values of the deflection at the boundary are prescribed. (For the clamped plate and cantilever, the normal derivatives of the deflection at the boundary are also preassigned). This necessitates the vanishing of the function  $h(x,y)$  (and possibly its normal derivatives) along the boundary of the given domain. However, in many important applications boundary values are not preassigned. Rather, the nature of the problem often allows complete or partial freedom for the boundary values of the system. In such cases, as we shall see, the stationary condition

$$\bar{V}[v, h] = 0$$

for the potential energy  $\bar{V}[v, v]$  leads automatically to boundary conditions for the corresponding boundary value problem. These we call the natural boundary conditions in contrast to the artificially fixed (or prescribed) boundary conditions with which, up to now, we have been concerned. Physically, the natural boundary conditions correspond to the freedom of motion of a system at its boundary. The general idea of the handling of problems of this nature is best grasped by considering a few examples.



a) The String: We consider a homogeneous string which is "elastically tied" at both ends by elastic forces of intensity  $h_0$  per unit displacement at  $x = 0$  and  $h_1$  at  $x = \ell$ . We seek the equilibrium position of the string under an external force of density  $f(x)$ . To do this we minimize the potential energy



$$(34) \quad \bar{V}[v, v] = \int_0^{\ell} (v'^2 + vf)dx + h_0 v(0)^2 + h_1 v(\ell)^2.$$

The class of admissible functions  $v(x) + \xi h(x)$  is quite arbitrary since no boundary conditions are imposed on  $h(x)$ . As usual, the stationary character of (34) is expressed by the vanishing of the polar form, thus

$$(35) \quad \bar{V}[v, h] = \int_0^{\ell} (v'h' + hf)dx + h_0 v(0)h(0) + h_1 v(\ell)h(\ell) = 0$$

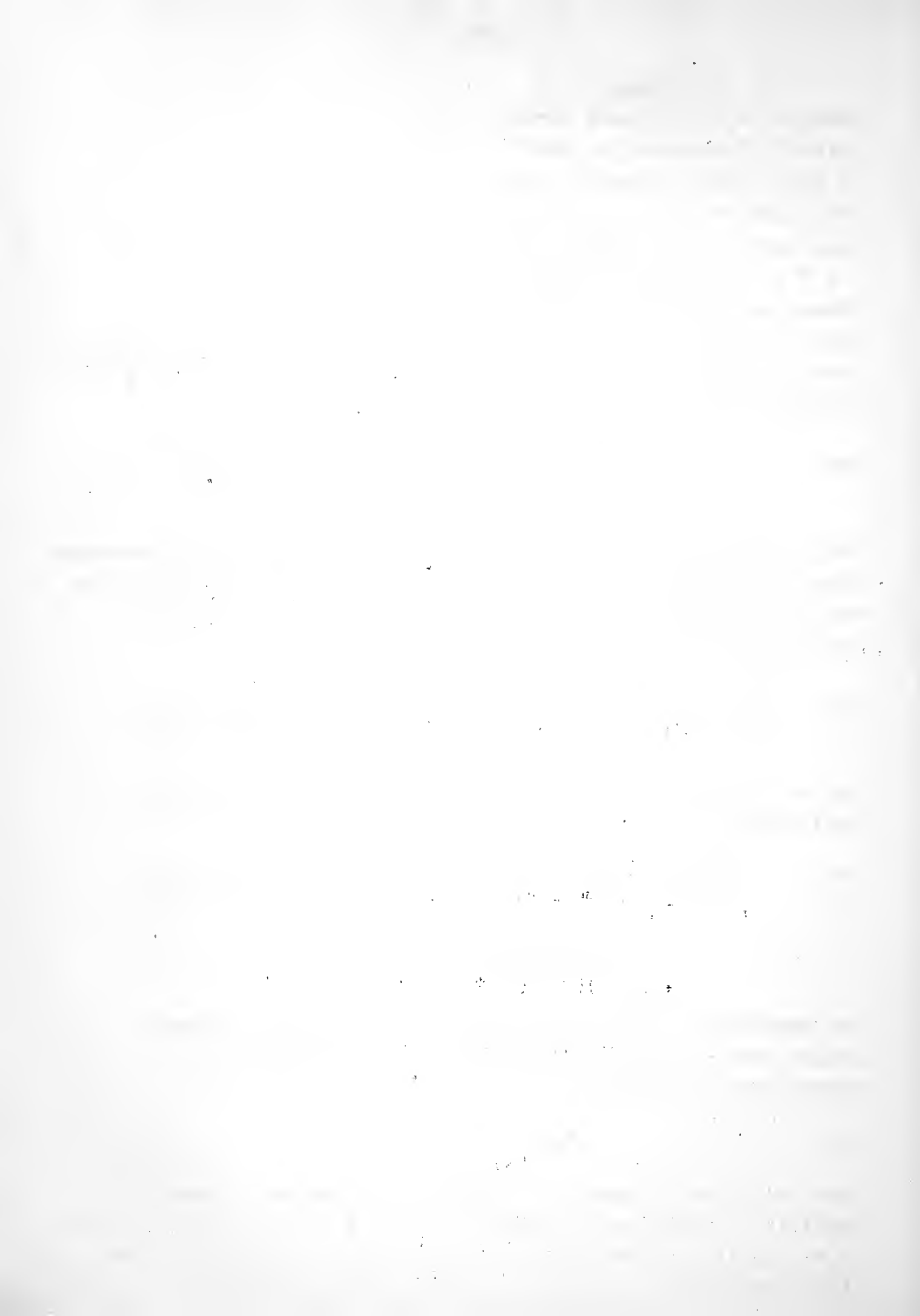
for every  $h$  of the class. Integrating by parts the integral of (35) yields

$$(35') \quad \bar{V}[v, h] = - \int_0^{\ell} h(v'' - f)dx - h(0)[v'(0) = h_0 v(0)] \\ + h(\ell)[v'(\ell) + h_1 v(\ell)] = 0.$$

The arbitrary character of  $h(x)$  enables us, in one stroke, to reduce (35') to a boundary value problem. First of all the differential equation

$$(36) \quad v''(x) = f(x)$$

must hold, for we need only consider the narrower class of admissible functions for which  $h(0) = h(\ell) = 0$  and obtain (36) as we did in the last section for the string with fixed ends. It follows that





$$-h(0)[v'(0) - h_0 v(0)] + h(\ell)[v'(\ell) + h_1 v(\ell)] = 0$$

for arbitrary  $h(x)$ . If we restrict ourselves to functions  $h(x)$  which vanish at  $x = 0$ , we conclude that

$$(37) \quad v'(\ell) + h_1 v(\ell) = 0 \quad .$$

Likewise, for  $h$  vanishing at  $x = \ell$ , we have

$$(38) \quad v'(0) - h_0 v(0) = 0 \quad .$$

Hence, we have obtained the required boundary value problem for the equilibrium of the "elastically tied" string, to wit, equation (36) and the natural boundary conditions (37) and (38).

Incidentally, we observe that, if the elastic forces are made to diminish, i.e., if  $h_0$  and  $h_1$  tend to zero, we obtain the case of a string with free boundaries. The natural boundary conditions automatically become

$$v'(0) = v'(\ell) = 0 \quad .$$

On the other hand, if the elastic forces are stiffened, then  $h_0$  and  $h_1$  increase beyond all bounds and we obtain the boundary conditions for a string with fixed ends, namely,

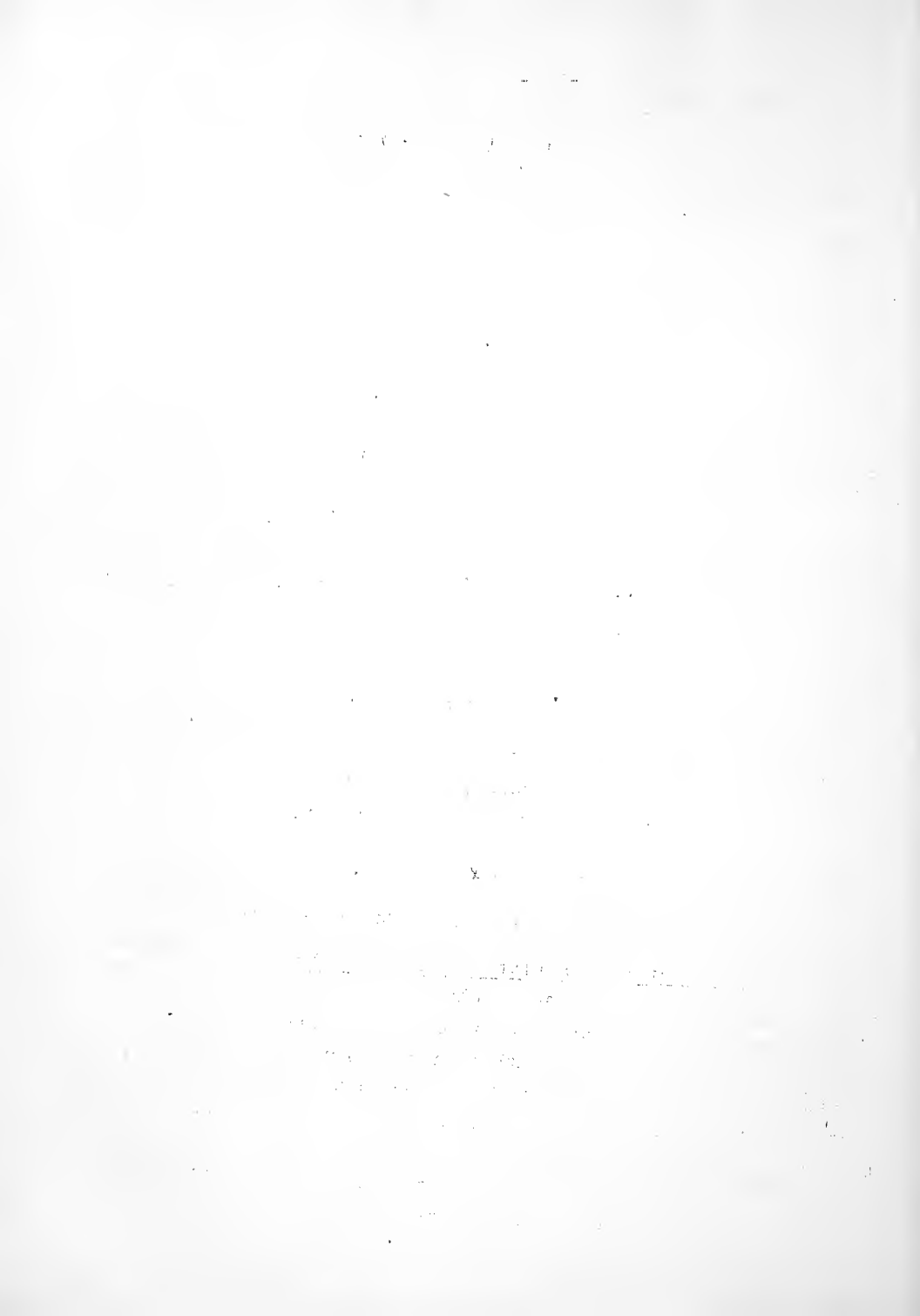
$$v(0) = v(\ell) = 0 \quad ,$$

since we require the potential energy (34) to remain finite.

#### b) The Rectangular Plate with Partially Free Boundaries:

In the theory of the bending of plates, the problem of determining the proper boundary conditions at the free boundaries had puzzled physicists and mathematicians for many decades. Eventually they were discovered by Kirchhoff. As a matter of fact, they are a very simple consequence of our variational theory.

As an example, consider the problem of the equilibrium of a square plate spanned over the region



$$\begin{cases} -1 \leq x \leq 1 \\ -1 \leq y \leq 1 \end{cases}$$

of the  $x, y$ -plane. We suppose that the edges of the plate along the lines  $x = -1$  and  $x = 1$  are clamped. That is

$$(39) \quad v = v_x = v_y = 0 \quad \text{for } x = \pm 1.$$

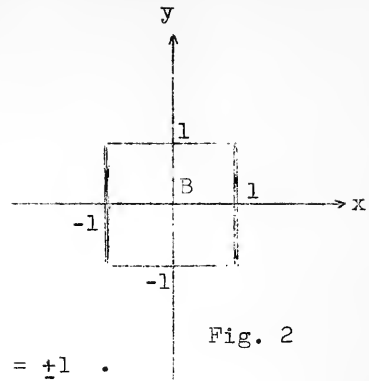


Fig. 2

The edges along the lines  $y = -1$  and  $y = 1$  we assume to be entirely free. The proper boundary conditions along these lines will be furnished by the variational theory when we minimize the potential energy

$$\bar{V}[v, v] = \int_B [(\nabla^2 v)^2 - v f] dx dy - 2(1 - \gamma) \int_B (v_{xx} v_{yy} - v_{xy}^2) dx dy$$

of the plate. As our admissible functions  $v(x, y) + \varepsilon h(x, y)$  we take those for which  $h$  satisfies the conditions (39) and is twice continuously differentiable but otherwise arbitrary. The stationary condition for  $\bar{V}$  is

$$\begin{aligned} (40) \quad \bar{V}[v, h] &= \int_B (\nabla^2 v \nabla^2 h + h f) dx dy \\ &\quad - (1 - \gamma) \int_B (v_{xx} h_{yy} + v_{yy} h_{xx} - 2v_{xy} h_{xy}) dx dy \\ &= 0. \end{aligned}$$

If we carry out in (40) the transformations indicated in the remark of the previous section, we obtain expression (31). However, since the domain  $B$  is rather simple in the present case, it may be well to carry out the calculations directly.

The first integral of (40) becomes, by Green's formula,



$$\int_B \int h(\nabla^2 \nabla^2 v + f) dx dy - \int_{-1}^1 \nabla^2 v h_y(x, -1) dx + \int_{-1}^1 \nabla^2 v h_y(x, 1) dx \\ - \int_{-1}^1 h(x, -1) \frac{\partial \nabla^2 v}{\partial y} dx + \int_{-1}^1 h(x, 1) \frac{\partial \nabla^2 v}{\partial y} dx \quad ,$$

by virtue of the conditions (39) imposed on  $h$ . The second integral of (40) may be transformed directly by integrating by parts. Carrying this out and using (39), we find that

$$\int_B \int (v_{xx} h_{yy} + v_{yy} h_{xx} - 2v_{xy} h_{xy}) dx dy \\ = - \int_{-1}^1 h_y(x, -1) v_{xx} dx + \int_{-1}^1 h_y(x, 1) v_{xx} dx \\ + \int_{-1}^1 h_x(x, -1) v_{xy} dx - \int_{-1}^1 h_x(x, 1) v_{xy} dx \quad .$$

The last two integrals on the right may be further simplified by integrating by parts and using the fact that  $h = 0$  at the four vertices of the square. Thus

$$\int_{-1}^1 h_x(x, -1) v_{xy} dx - \int_{-1}^1 h_x(x, 1) v_{xy} dx \\ = - \int_{-1}^1 h(x, -1) v_{xxy} dx + \int_{-1}^1 h(x, 1) v_{xxy} dx \quad .$$

Hence

$$\bar{V}[v, h] = \int_B \int h(\nabla^2 \nabla^2 v + f) dx dy - \int_{-1}^1 h_y(x, -1) (\gamma v_{xx} + v_{yy}) dx \\ + \int_{-1}^1 h_y(x, 1) (\gamma v_{xx} + v_{yy}) dx - \int_{-1}^1 h(x, -1) \left[ \frac{\partial \nabla^2 v}{\partial y} - (1 - \gamma) v_{xxy} \right] dx \\ - \int_{-1}^1 h(x, 1) \left[ \frac{\partial \nabla^2 v}{\partial y} - (1 - \gamma) v_{xxy} \right] dx \quad .$$



Now, by suitably restricting the class of admissible functions, we obtain the differential equation

$$(41) \quad \nabla^2 \nabla^2 v + f = 0$$

and the natural boundary conditions

$$(42) \quad \gamma v_{xx} + v_{yy} = 0 \quad \text{for } y = \pm 1,$$

$$(42') \quad \frac{\partial}{\partial y} (\gamma v_{xx} + v_{yy}) = 0 \quad \text{for } y = \pm 1.$$

c) The Reinforced Plate: Exceedingly important is the problem of the equilibrium of a plate reinforced by elements possessing other elastic or bending properties. For example, we may imagine bars or cantilevers attached along certain lines of a plate. Along such lines the elastic or bending forces are different from those of the plate and we are led to certain discontinuity conditions for the derivatives of the deflection of the plate. Rather than proceeding with the general variational theory of such problems, we again consider an example which gives a sufficient indication of the general procedure.

Consider again the square plate of the previous example.

Suppose now, however, that it is clamped along its entire boundary so that

$$v = v_x = v_y = 0$$

$$\text{for } x = \pm 1 \text{ and } y = \pm 1.$$

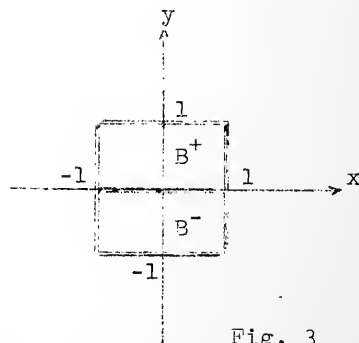
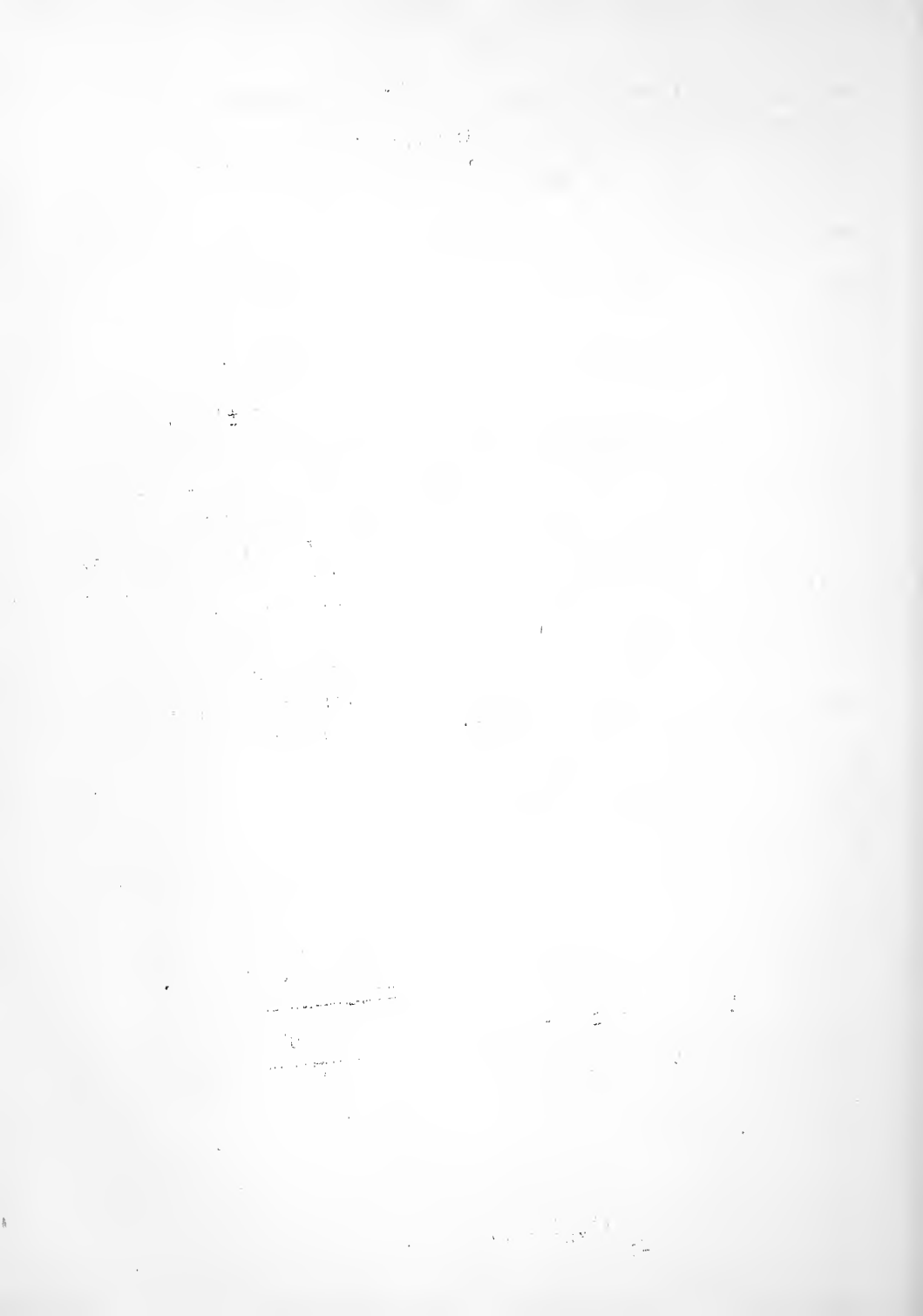


Fig. 3

Furthermore, assume that along the line  $y = 0$  the plate is reinforced by a bar whose energy is given by

$$\int_{-1}^1 (v_{xx}(x,0))^2 dx.$$





The total energy of the compound system is, then,

$$(43) \quad \bar{V}[v,v] = Q[v,v] + \int_B \int_0^1 v f \, dx \, dy + \int_{-1}^1 (v_{xx}(x,0))^2 dx$$

where  $Q[v,v]$  is given by (5). Physically, it is plausible that the required equilibrium position  $v(x,y)$  will possess a discontinuity in at least one of its derivatives along the line  $y = 0$ , since the bar has properties different from those of the plate. Furthermore, this discontinuity will appear only in a derivative along the normal to this line, i.e., in a derivative with respect to  $y$ . (We must forego the mathematical justification of these facts and also of the fact that the discontinuity cannot appear in the first or second derivatives, i.e., in the tangent plane or curvature of the surface  $v = v(x,y)$ ).

The discontinuity condition is easily obtained. We set up the stationary condition for (43), thus

$$\bar{V}[v,h] = Q[v,h] + \int_B \int_0^1 h f \, dx \, dy + \int_{-1}^1 v_{xx} h_{xx} dx = 0 \quad .$$

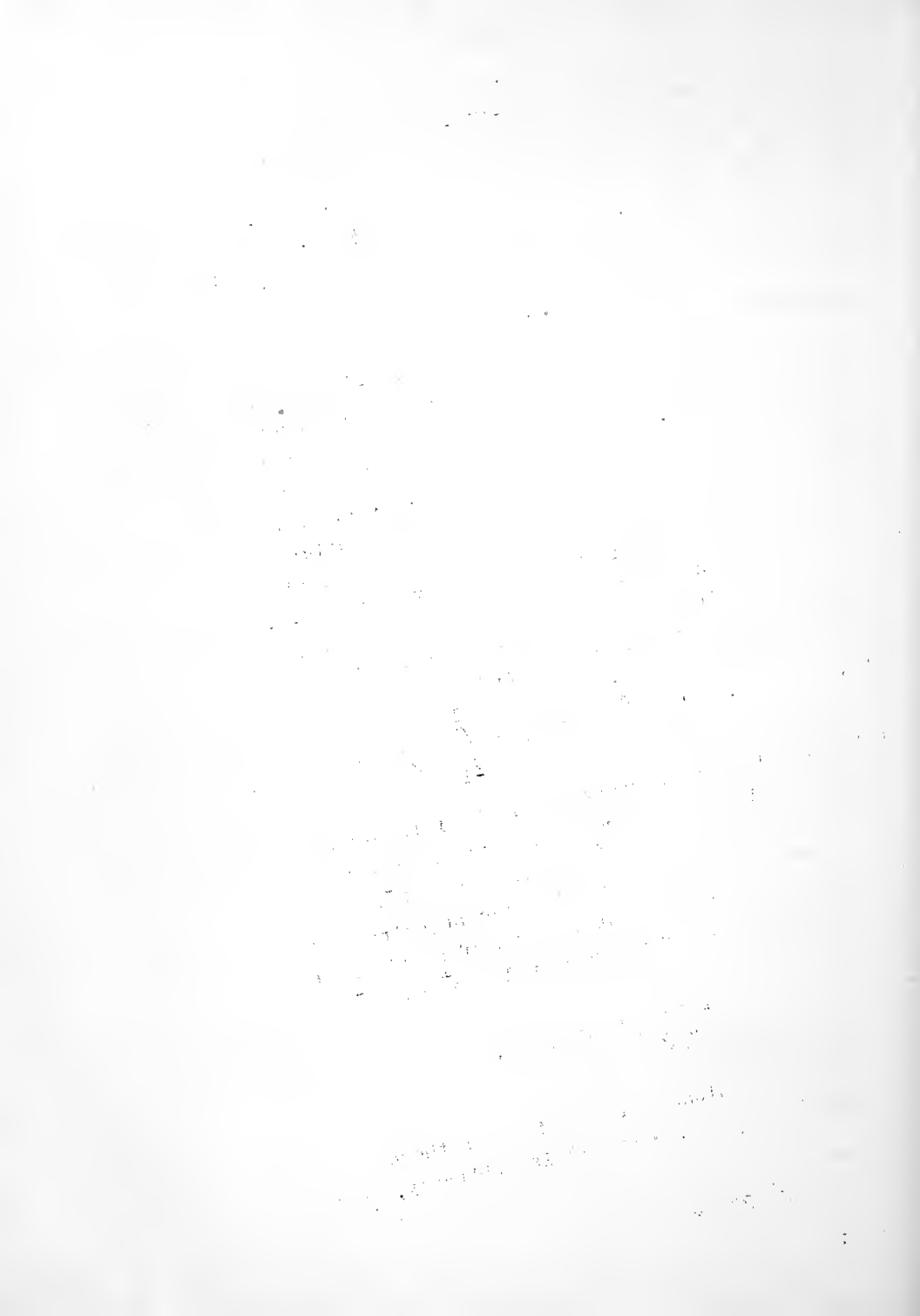
In order to effect a transformation of the integral expression  $\bar{V}[v,h]$  we must divide the square  $B$  into two rectangles by the line  $y = 0$ ; the upper we call  $B^+$  and the lower  $B^-$ . (See figure 3). This is necessary because of the discontinuity we know will appear along  $y = 0$ . Carrying out the transformations of the preceding example in each of the rectangles  $B^+$  and  $B^-$  for the expression

$$Q[v,h] + \int_B \int_0^1 h f \, dx \, dy \quad ,$$

and remembering that  $h$  vanishes on all sides of these rectangles except along  $y = 0$ , where it is arbitrary, we find that

$$(44) \quad \nabla^2 \nabla^2 v + f = 0$$

except along  $y = 0$ ; also



$$(45) \quad \int_{-1}^1 h_{xx} v_{xx} dx + \int_{-1}^1 h_y [\varphi(x, 0-) - \varphi(x, 0+)] dx \\ + \int_{-1}^1 h [\psi_y(x, 0-) - \psi_y(x, 0+)] dx = 0 \quad .$$

Here  $\varphi(x, y) = \gamma v_{xx} + v_{yy}$ , while  $\varphi(x, 0+)$  and  $\varphi(x, 0-)$  represent the values of  $\varphi$  on  $y = 0$  obtained by approaching this line from  $B^+$  and  $B^-$ , respectively. We may simplify (45) still further by assuming the curvature of the plate to be continuous. On this assumption, the second integral in (45) vanishes since  $\varphi(x, 0-) = \varphi(x, 0+)$ . Likewise, by preceding remarks,

$$\int_{-1}^1 h_{xx} v_{xx} dx = \int_{-1}^1 h v_{xxxx} dx \quad .$$

Thus (45) becomes

$$\int_{-1}^1 h \{ v_{xxxx} - [\psi_y(x, 0+) - \psi_y(x, 0-)] \} dx = 0$$

for arbitrary  $h$ . Immediately we obtain the required discontinuity condition

$$\psi_y(x, 0+) - \psi_y(x, 0-) = v_{xxxx}(x, 0) \quad ,$$

or

$$(46) \quad \left. \frac{\partial}{\partial y} (\gamma v_{xx} + v_{yy}) \right|_{y=0+} - \left. \frac{\partial}{\partial y} (\gamma v_{xx} + v_{yy}) \right|_{y=0-} = v_{xxxx}(x, 0) \quad .$$

Thus, a certain combination of the third derivatives of the deflection of the plate suffers a discontinuity of amount  $v_{xxxx}(x, 0)$  along the line of reinforcement  $y = 0$ .

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

$$f''(x) = \frac{6}{x^4}$$

$$f'''(x) = -\frac{24}{x^5}$$

$$f^{(4)}(x) = \frac{120}{x^6}$$

$$f^{(5)}(x) = -\frac{720}{x^7}$$

$$f^{(6)}(x) = \frac{5040}{x^8}$$

$$f^{(7)}(x) = -\frac{40320}{x^9}$$

$$f^{(8)}(x) = \frac{322560}{x^{10}}$$

$$f^{(9)}(x) = -\frac{2580480}{x^{11}}$$

$$f^{(10)}(x) = \frac{20643840}{x^{12}}$$

$$f^{(11)}(x) = -\frac{165150720}{x^{13}}$$

#### 4. Remarks Concerning Approximation of Functions

In connection with the theory of eigen-functions and the generalized Fourier series of Chapter Two we encountered the concept of a complete function system. In the applications of the Calculus of Variations to boundary value and vibration problems we often find it necessary to approximate an arbitrary function  $w(x)$  or  $w(x,y)$  by a linear combination

$$(47) \quad \bar{w} = \sum_{i=1}^n c_i \omega_i$$

of rather simple coordinate functions

$$(48) \quad \omega_1, \omega_2, \omega_3, \dots, \omega_n, \dots$$

given or chosen in advance. If the coordinate functions form a complete system we know that the approximating function  $\bar{w}$  may be chosen to any degree of accuracy, at least in the sense of a mean square deviation

$$\int_B (w - \bar{w})^2 dx dy ,$$

provided only that the number of coordinate functions used is sufficiently large. Here we shall concern ourselves with function systems which are complete in a stricter sense, i.e., in the sense that the linear measure of approximation  $|w - \bar{w}|$  may be made arbitrarily small.

Except for the trigonometric functions, the most important and most useful coordinate functions are the integral powers of  $x$ , or, in two dimensions,  $x^n y^m$ . The linear combinations of such coordinate functions are polynomials

$$P_n(x) = \sum_{i=0}^n c_i x^i$$

and

$$P_n(x,y) = \sum_{i,j=0}^n c_{ij} x^i y^j .$$

Weierstrass proved the following important theorem:

1. 1. 1.

2. 2. 2.

3. 3. 3.

(1)  
(2)  
(3)

4. 4. 4.

5. 5. 5.

6. 6. 6.

If  $f(x)$  is an arbitrary continuous function in the closed interval  $a \leq x \leq b$ , then it may be approximated in this interval to any desired degree of accuracy by a polynomial  $P_n(x)$ , provided that  $n$  is taken sufficiently large.

This theorem is valid for higher dimensions as well. E.g., such an approximation can be found for any continuous function in a closed rectangle. Moreover, it can be shown that, if  $f(x)$  is not only continuous but has, as well, continuous derivatives up to the  $k^{\text{th}}$  order, then a polynomial can be found which not only approximates  $f(x)$  but also is such that its derivatives up to the  $k^{\text{th}}$  order approximate the corresponding derivatives of  $f(x)$ .

In conclusion, we make an incidental but interesting remark of a mathematical character. C. H. Munz proved that, in a finite closed interval, not only the functions

$$(49) \quad 1, x, x^2, x^3, \dots, x^n, \dots$$

but, more generally, the functions

$$(50) \quad 1, x^{r_1}, x^{r_2}, \dots, x^{r_n}, \dots$$

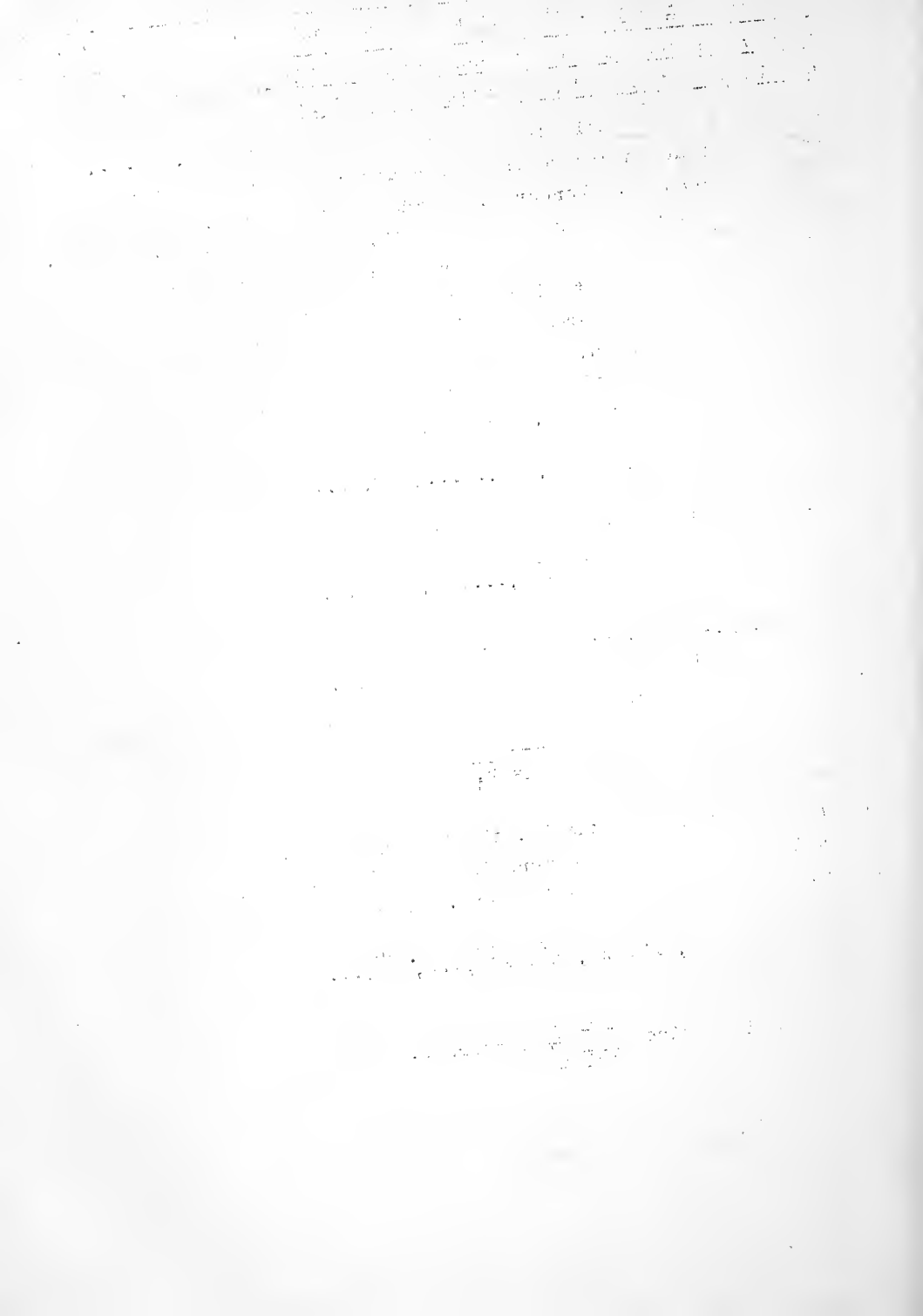
where  $r_1, r_2, \dots, r_n, \dots$  is any sequence of positive real numbers, form a complete system in the strict linear sense if and only if the infinite sum

$$\sum_{i=1}^{\infty} \frac{1}{r_i}$$

diverges. Thus, for example, the subsequence of (49) obtained by taking only the even powers is a complete system in the strict sense, and likewise for odd powers. However, the subsequence

$$1, x^2, x^4, x^8, x^{16}, \dots, x^{2^n}, \dots$$

is not complete since  $\sum_{i=1}^{\infty} \frac{1}{2^i}$  converges.





### Part III. The Rayleigh-Ritz Method

#### 1. Equilibrium Problems

Lord Rayleigh and, independently but from a more general viewpoint, Walter Ritz were very successful, from a practical standpoint, in applying variational methods to problems of equilibrium. We have seen that the variational theory reduces a minimum problem, such as that for the equilibrium of a membrane, to a boundary value problem for a differential equation. Such a reduction is of little practical value in solving the variational problem. Rather, we consider the reverse procedure. Instead of attacking the boundary value problem as a means of solving the variational problem we work with the variational problem to the end of finding a solution to a given boundary value problem. The ideas and principles of the direct variational calculus and, in particular, of the Rayleigh-Ritz method, will be considered in detail in a later course. At this time it is possible only to state the principles involved and to illustrate them by a few examples.

First of all, we remark that we cannot hope to solve a given variational problem at one stroke; instead, we seek to find an approximate solution by means of the following steps:

a) We choose a system of admissible coordinate functions

$$(1) \quad \{\omega_i\} = \omega_1, \omega_2, \dots, \omega_n, \dots$$

which are such that any linear combination

$$(2) \quad \varphi_n = \sum_{i=1}^n c_i \omega_i$$

is again an admissible function in our variational problem, the  $c_i$  being arbitrary constants.

b) We substitute  $\varphi_n$  in the given variational problem causing it to degenerate into an ordinary minimum problem for the  $n$  constants  $c_i$ . This new minimum problem, according to the rules of the calculus, leads to a system of  $n$  equations in  $n$  unknowns. By solving this system we obtain those values of the



$c_1$  which yields the best possible  $\varphi_n$ . For example, if the given variational problem consists in minimizing a quadratic expression  $Q[v, v]$ , where  $v$  satisfies certain prescribed boundary conditions, then the equations for the  $c_1$  are linear:

$$(3) \quad \frac{\partial}{\partial c_1} Q[\varphi_n, \varphi_n] = \frac{\partial}{\partial c_1} Q\left[\sum_{j=1}^n c_j \omega_j, \sum_{j=1}^n c_j \omega_j\right] = 0 \quad , \quad (i = 1, \dots, n) \quad .$$

According to the rules of section 1 of Part II, (3) becomes

$$(4) \quad \sum_{j=1}^n c_j Q[\omega_i, \omega_j] = 0 \quad , \quad (i = 1, \dots, n) \quad .$$

c) Finally, the function  $\varphi_n$ , for which the  $c_i$  are given by (4), is considered to be (and, in many cases, can actually be proved to be) a "good" approximation to the actual solution  $v(x, y)$  of the variational problem.

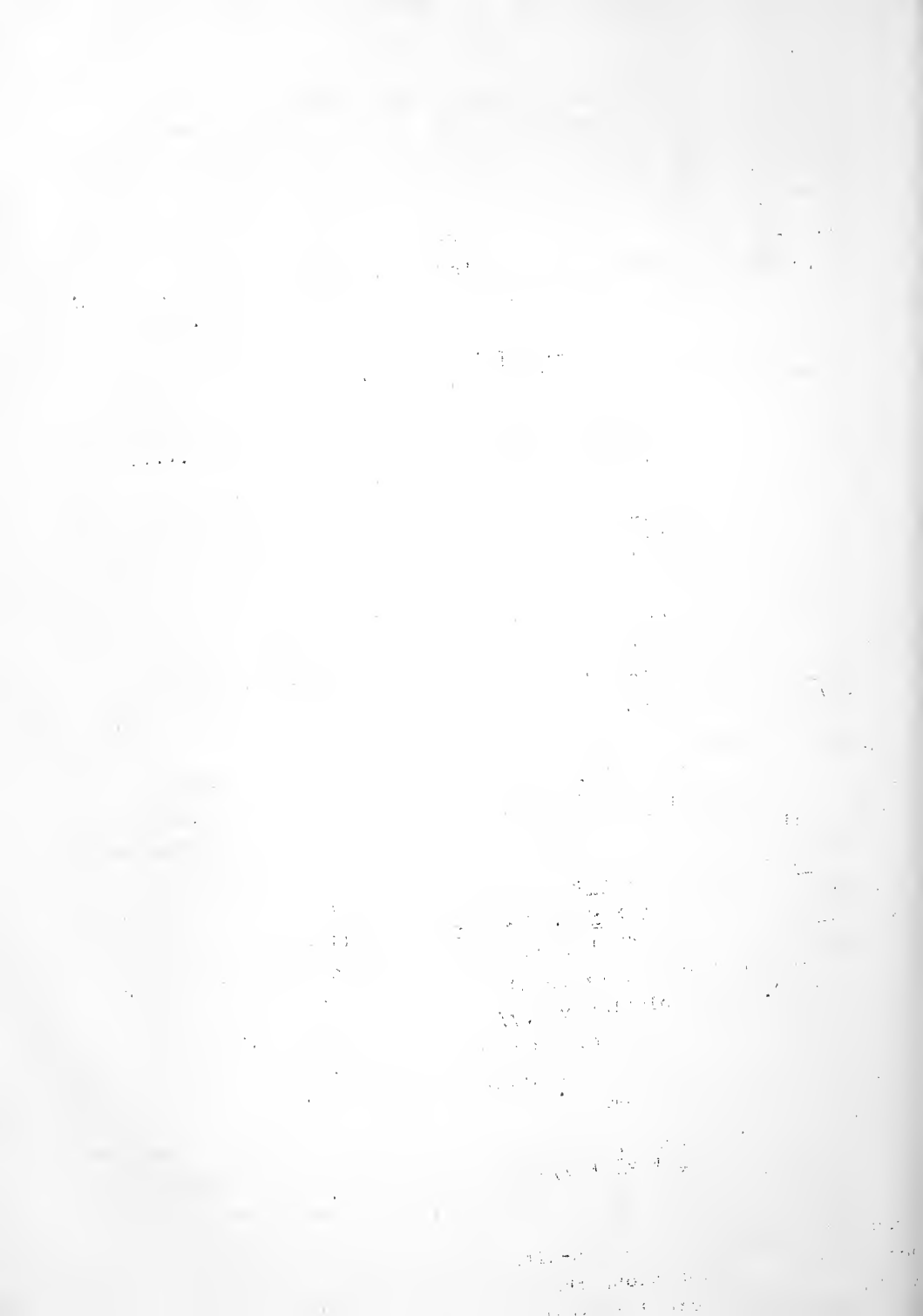
The mathematical theory and justification is extensive and not altogether simple. As we have mentioned already, we leave this to a later course and content ourselves here with some special examples.

The Square Membrane: Consider a membrane stretched over a square domain  $B$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , and fixed along the boundary. We seek to find approximately the equilibrium position of the membrane under an external force of density one ( $f(x, y) = 1$ ). The solution  $v(x, y)$  we know satisfies  $\nabla^2 v = 0$  in  $B$  and  $v = 0$  along the boundary of  $B$ .

The first step in the solution is to replace this problem by the variational problem

$$(5) \quad \int_B (v_x^2 + v_y^2 + v) dx \, dy = \text{minimum} \quad .$$

In order to apply the Rayleigh-Ritz method we choose coordinate functions which vanish along the boundary of  $B$ . Suitable functions are constructed by multiplying powers of  $x$  and  $y$  by the polynomial



$$(6) \quad P(x,y) = x(1-x)y(1-y) \quad .$$

We construct the admissible function

$$(7) \quad \varphi_1 = P(x,y)(c_1x + c_2y)$$

and substitute it into (5). The latter then becomes a function of the constants  $c_1$  and  $c_2$ , which when differentiated with respect to  $c_1$  and  $c_2$  leads to the two algebraic equations

$$\begin{cases} 96c_1 + 245c_2 = -175 \\ 245c_1 + 96c_2 = -175 \end{cases}$$

whose solution

$$c_1 = c_2 = -\frac{175}{341}$$

represents the values of  $c_1$  and  $c_2$  for which the minimum of (5) is attained for all functions  $\varphi_1$  of the form (7). The function  $\varphi_1$ , with  $c_1$  and  $c_2$  as evaluated, we accept as a good approximation to the actual solution  $v(x,y)$ .

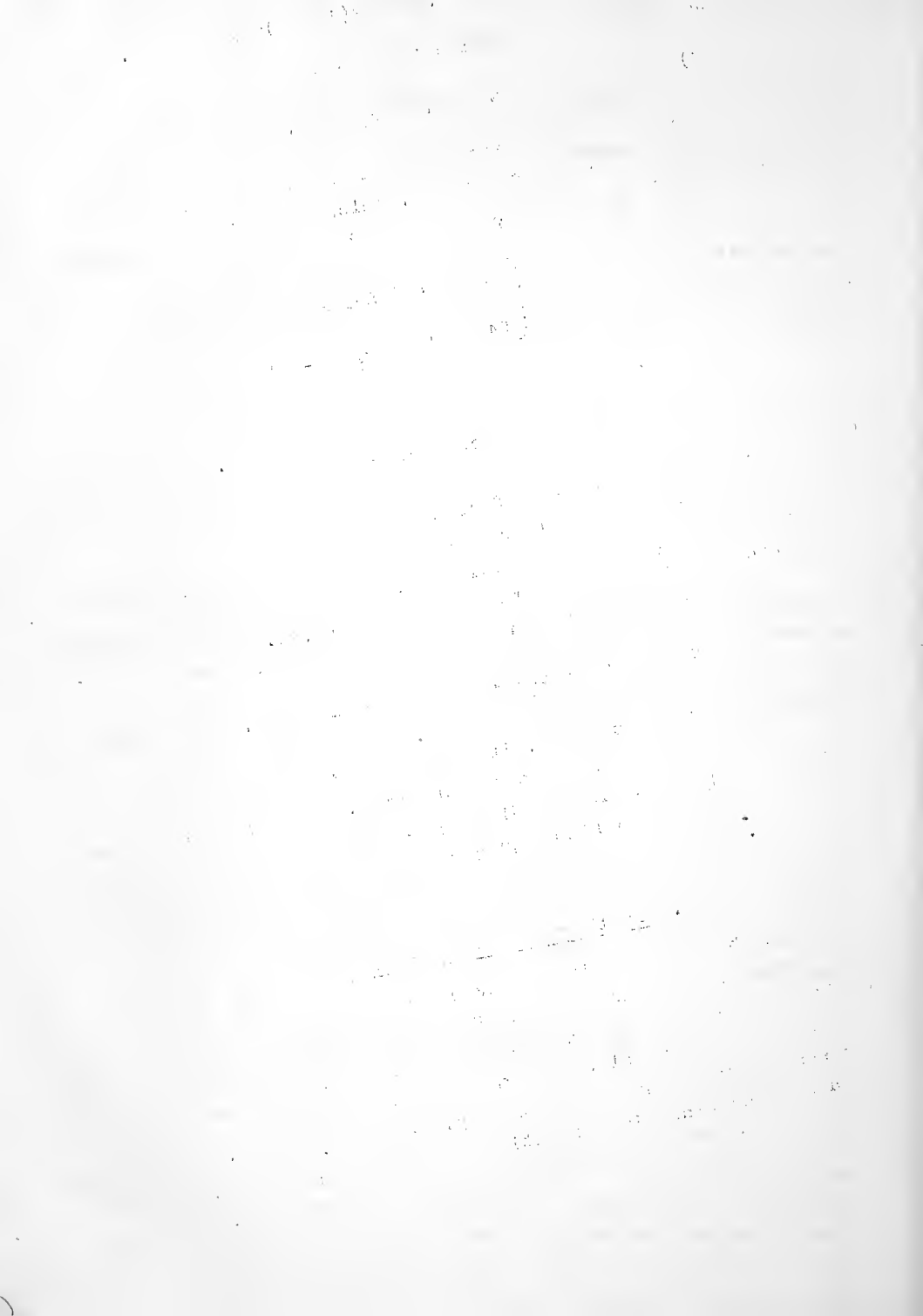
To obtain a better approximation we consider the function

$$\varphi_2 = P(c_0 + c_1x + c_2y + c_3x^2 + c_4y^2 + c_5xy) \quad .$$

The use of this function, it will be noticed, leads to a system of six linear equations in six unknowns. The numerical solution of these equations is a tedious job. The task of the applied mathematician is to find a proper compromise between labor and accuracy.

## 2. Further Examples and Remarks

In the case of fixed boundary conditions only a few terms are needed in the construction of the function  $\varphi$  in order to obtain a sufficiently good approximation, while in problems involving free boundaries many more terms are necessary in order to procure a like degree of accuracy. However, in the latter case a compensating simplification appears in that the choice



of coordinate functions is less restricted since no prescribed boundary conditions need be satisfied. Such a simplification is of practical importance and, hence, of great value is the following device when applied to problem with fixed boundaries.

Suppose we are considering a problem in which the boundaries are fixed and in which the domain B is not a simple domain such as a circle or rectangle. In general, it will be quite difficult to find simple coordinate functions satisfying the boundary conditions. This difficulty may be overcome, however, if we treat the problem with fixed boundaries as a limiting case of a problem with free boundaries. An example will best demonstrate the usefulness of this device.

Let an elastic membrane be stretched over a general domain B of the x,y-plane. The problem is to find the equilibrium position of this membrane if it is subjected to an external force of density  $f(x,y)$  and if it is fixed along the boundary C of B. In order to obtain an approximate solution to this problem we consider another, slightly different, problem. Let the membrane be not fixed along C but, instead, restrained there by a strong elastic force of density  $\sigma(s)$ . According to our general pattern, this problem is equivalent to the variational problem

$$(8) \quad \int_B (v_x^2 + v_y^2 + vf) dx dy + \int_C \sigma v^2 ds = \text{minimum} .$$

Since no artificial boundary conditions are imposed upon the admissible functions for (8), we may choose rather arbitrary coordinate functions. This freedom of choice makes it quite simple to find an approximation to the actual solution of (8). Now, it is known from the earlier part of this chapter that the resulting free boundary condition for (8), obtained by the variational method, is

$$(9) \quad \frac{\partial v}{\partial n} + \sigma v = 0 \quad \text{on } C .$$

It is clear, physically, that if the elastic restoring force along C is very strong, i.e., if  $\sigma$  is large, then the deflection  $v$  at the boundary must be correspondingly small in order that





the potential energy given by (8) remain finite. Thus, if  $\sigma$  is allowed to increase beyond all bounds, the deflection  $v$  must tend to zero on  $C$ , so that the condition (9) becomes

$$(10) \quad v = 0 \quad \text{on } C, \quad ,$$

which is the condition for a membrane with a fixed boundary, the problem which we originally set out to solve. It is in this sense that fixed boundary conditions are limiting cases of free boundary conditions. Furthermore, it is plausible (and a rigorous proof can be given) that the solution of the free boundary problem approaches that of the fixed boundary problem when the limiting process is carried out. It is this fact that is the kernel of the applicability of this device to the obtaining of numerical solutions. One further remark may be made. Though the approximation to the solution of the fixed boundary problem becomes better as  $\sigma$  increases, it is also true that the task of performing the calculations becomes more laborious. Again, it is the task of the practical mathematician to find the proper compromise between accuracy and labor. Actually, there is much research yet to be done in order to find the proper theoretical basis for the optimal choice of the function  $\sigma$ .

It might be well to mention that the great success attained by Ritz in the theory of plates was later found to be due, to a great extent, to the fact that the differential equation for a plate is of the fourth order. It is because of this that a rather smooth and fast convergence of the approximating solutions to the actual solution is possible. On the other hand, in the case of the membrane, where the differential equation is of the second order, the situation is quite different. In general, we may not even assume that the approximating solutions will converge at all to the actual solution. The theoretical basis for these facts must be omitted here. It may be pointed out, however, that, in general, the higher the order of the differential equation, the greater are the number of terms necessary for a prescribed degree of accuracy.



These theoretical remarks may also be used to advantage for the improvement of our previous approximation methods. For example, consider again the problem of equilibrium of the membrane with fixed boundary. Instead of considering the usual variational problem

$$(11) \quad \int_B \int (v_x^2 + v_y^2 + vf) dx dy = \text{minimum} \quad ,$$

we consider the apparently more complicated one

$$(12) \quad \int_B \int (v_x^2 + v_y^2 + vf) dx dy + \mu \int_B \int (\nabla^2 v - f) dx dy = \text{minimum} \quad .$$

Since a condition for the minimum of (12) is obviously  $\nabla^2 v - f = 0$ , it is clear that the minima for (11) and (12) are the same. Though problem (12) seems much more difficult because it involves more derivatives, yet, in accord with our preceding remarks, it offers a better guarantee for the convergence of the approximating solutions  $\varphi_1, \varphi_2, \dots$ . The factor  $\mu$  in (12) is at our disposal for a suitable compromise. The theoretical and practical basis for this method remains an important field for further investigations.

### 3. Applications to the Theory of Vibrations

The variational methods are perhaps even more important for vibration problems than for problems of equilibrium. The characterization of eigen-values as minima and maxi-minima of the quotient  $\bar{V}/\bar{T}$  of the reduced potential and kinetic energies carries over from systems of a finite number of degrees of freedom to continuous systems such as strings, membranes and plates. The problems of vibrations of continuous systems lead to the following general type of variational problem.

Given homogeneous boundary conditions or no boundary conditions at all, the procedure is to minimize a certain quadratic expression  $A[v, v]$ , the reduced potential energy, under the condition that another quadratic expression  $H[v, v]$ , the reduced kinetic energy, remain equal to unity. The value  $\lambda_1$



of this minimum is the first eigen-value of the system and the corresponding minimizing function  $v_1$  is the first eigen-function. The other eigen-values and eigen-functions of the system are obtained successively as follows.

The  $k^{\text{th}}$  eigen-value  $\lambda_k$  is the minimum of  $Q[v,v]$  under the "normalization" condition  $H[v,v] = 1$  and the  $k-1$  "orthogonal" conditions

$$(13) \quad H[v, v_i] = 0 \quad , \quad (i = 1, 2, \dots, k-1) \quad ,$$

where  $v_i$  is the  $i^{\text{th}}$  eigen-function of the system.

We must forego the proof of these statements in this course. However, we briefly indicate some of the reasoning in connection with the special case of a membrane with fixed boundary. In this case

$$(14) \quad \frac{\bar{V}}{T} = \frac{Q[v,v]}{H[v,v]} = \frac{\int_B (v_x^2 + v_y^2) dx \, dy}{\int_B v^2 dx \, dy}$$

is to be minimized with respect to all admissible functions vanishing along the boundary  $C$  of  $B$ . Evidently, as we have already seen for systems of a finite number of degrees of freedom, under proper normalization the minimum of (14) is the same as the minimum of  $Q[v,v]$  under the condition that  $H[v,v] = 1$ . If  $\lambda_1$  is the minimum of (14) and  $v_1$  the corresponding minimizing function, it follows that, for any admissible function  $v_1 + \varepsilon h = w$ ,

$$(15) \quad Q[w,w] \geq \lambda_1 H[w,w] \quad .$$

Using the expansion rule of page 160 and the facts that  $Q[v_1, v_1] = \lambda_1$  and  $H[v_1, v_1] = 1$ , (15) becomes

$$(16) \quad [2(Q[v_1, h] - \lambda_1 H[v_1, h]) + \varepsilon (Q[h, h] - \lambda_1 H[h, h])] \geq 0$$

for all  $\varepsilon$  and  $h$ . However, a simple investigation reveals that if (16) is to be true for all values of  $\varepsilon$ , then necessarily



$$(17) \quad Q[v_1, h] - \lambda_1 H[v_1, h] = 0$$

for arbitrary permissible  $h$ . Finally, by using the explicit expressions for  $Q$  and  $H$  as in (14) and reducing the variational condition (17) to a boundary value problem according to previous patterns, it follows that

$$(18) \quad \nabla^2 v_1 + \lambda_1 v_1 = 0$$

with  $v_1 = 0$  on  $C$ . By definition it is immediate that  $\lambda_1$  is an eigen-value of the fixed membrane and  $v_1$  the corresponding eigen-function. The fact that  $\lambda_1$  is the first eigen-value cannot be proved here but it is at least plausible since  $\lambda_1$  is the smallest value of  $\bar{V}/T$ . By using the conditions (13) the proof for higher eigen-values follows similarly.

Just as in the case of systems with a finite number of degrees of freedom, we may divorce the definition of the higher eigen-values from a knowledge of the preceding ones. This is accomplished by replacing minimum problems by problems of maxima. This independent definition follows.

Let  $w_1, w_2, \dots, w_{k-1}$  be any  $k-1$  admissible functions and let  $d(w_1, \dots, w_{k-1})$  be the minimum of  $Q[v, v]$  under the conditions  $H[v, v] = 1$  and

$$(19) \quad H[v, w_i] = 0 \quad , \quad (i = 1, 2, \dots, k-1) .$$

The  $k^{\text{th}}$  eigen-value  $\lambda_k$  of the corresponding continuous system is then the maximum of the minima  $d(w_1, \dots, w_{k-1})$  as the  $w$ 's range over all admissible functions. In fact, this maximum is attained when  $w_i = v_i$  ( $i = 1, \dots, k-1$ ), where  $v_i$  is the  $i^{\text{th}}$  eigen-function.

According as to the specific expression  $Q[v, v]$  this general scheme covers all types of vibrations discussed in this course. The theorems concerning systems of finite degrees of freedom in Chapter One carry over to continuous systems very easily. However, we do not intend to go into detail here. Rather, we give a short indication of how to employ the Rayleigh-Ritz method for obtaining numerical results in the theory of vibrations of continuous systems.





$$(17) \quad Q[v_1, h] - \lambda_1 H[v_1, h] = 0$$

for arbitrary permissible  $h$ . Finally, by using the explicit expressions for  $Q$  and  $H$  as in (14) and reducing the variational condition (17) to a boundary value problem according to previous patterns, it follows that

$$(18) \quad \nabla^2 v_1 + \lambda_1 v_1 = 0$$

with  $v_1 = 0$  on  $C$ . By definition it is immediate that  $\lambda_1$  is an eigen-value of the fixed membrane and  $v_1$  the corresponding eigen-function. The fact that  $\lambda_1$  is the first eigen-value cannot be proved here but it is at least plausible since  $\lambda_1$  is the smallest value of  $\bar{V}/T$ . By using the conditions (13) the proof for higher eigen-values follows similarly.

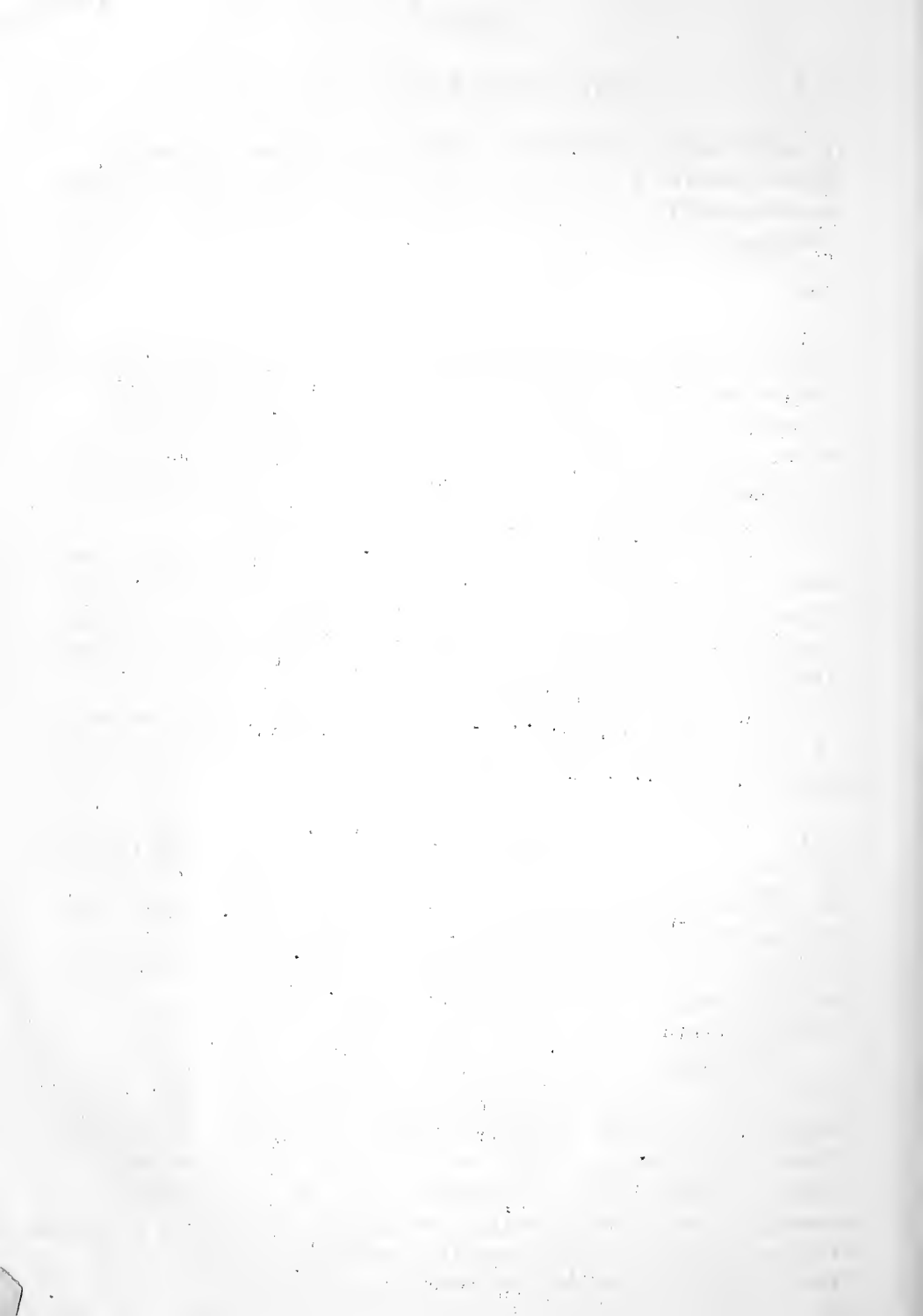
Just as in the case of systems with a finite number of degrees of freedom, we may divorce the definition of the higher eigen-values from a knowledge of the preceding ones. This is accomplished by replacing minimum problems by problems of maxima-minima. This independent definition follows.

Let  $w_1, w_2, \dots, w_{k-1}$  be any  $k-1$  admissible functions and let  $d(w_1, \dots, w_{k-1})$  be the minimum of  $Q[v, v]$  under the conditions  $H[v, v] = 1$  and

$$(19) \quad H[v, w_i] = 0 \quad , \quad (i = 1, 2, \dots, k-1) .$$

The  $k^{\text{th}}$  eigen-value  $\lambda_k$  of the corresponding continuous system is then the maximum of the minima  $d(w_1, \dots, w_{k-1})$  as the  $w$ 's range over all admissible functions. In fact, this maximum is attained when  $w_i = v_i$  ( $i = 1, \dots, k-1$ ), where  $v_i$  is the  $i^{\text{th}}$  eigen-function.

According as to the specific expression  $Q[v, v]$  this general scheme covers all types of vibrations discussed in this course. The theorems concerning systems of finite degrees of freedom in Chapter One carry over to continuous systems very easily. However, we do not intend to go into detail here. Rather, we give a short indication of how to employ the Rayleigh-Ritz method for obtaining numerical results in the theory of vibrations of continuous systems.



Most important among vibration problems is that of finding the fundamental frequency  $\lambda_1$  and the shape of the fundamental mode  $v_1$ . To the end of solving this problem we choose a system of coordinate functions  $\omega_i$  satisfying prescribed boundary conditions. We then substitute the linear combination

$$\varphi_n = \sum_{i=1}^n c_i \omega_i$$

in the variational expression so that  $Q$  and  $H$  become homogeneous expressions in the constants  $c_i$ , namely,

$$(20) \quad Q[\varphi_n, \varphi_n] = \sum_{i,j=1}^n c_i c_j Q[\omega_i, \omega_j] \quad ,$$

$$(20') \quad H[\varphi_n, \varphi_n] = \sum_{i,j=1}^n c_i c_j H[\omega_i, \omega_j] \quad .$$

We now determine the minimum of  $Q[\varphi_n, \varphi_n]$  with respect to the  $c_i$  under the condition  $H[\varphi_n, \varphi_n] = 1$ . This is exactly the type of problem discussed in part III of Chapter One. We are led to the algebraic equations

$$(21) \quad \sum_{j=1}^n c_j Q[\omega_i, \omega_j] - \lambda c_j H[\omega_i, \omega_j] = 0 \quad , \quad (i = 1, \dots, n) \quad ,$$

for the  $c$ 's. The values of  $\lambda$  are then obtained as the roots of the determinant equation

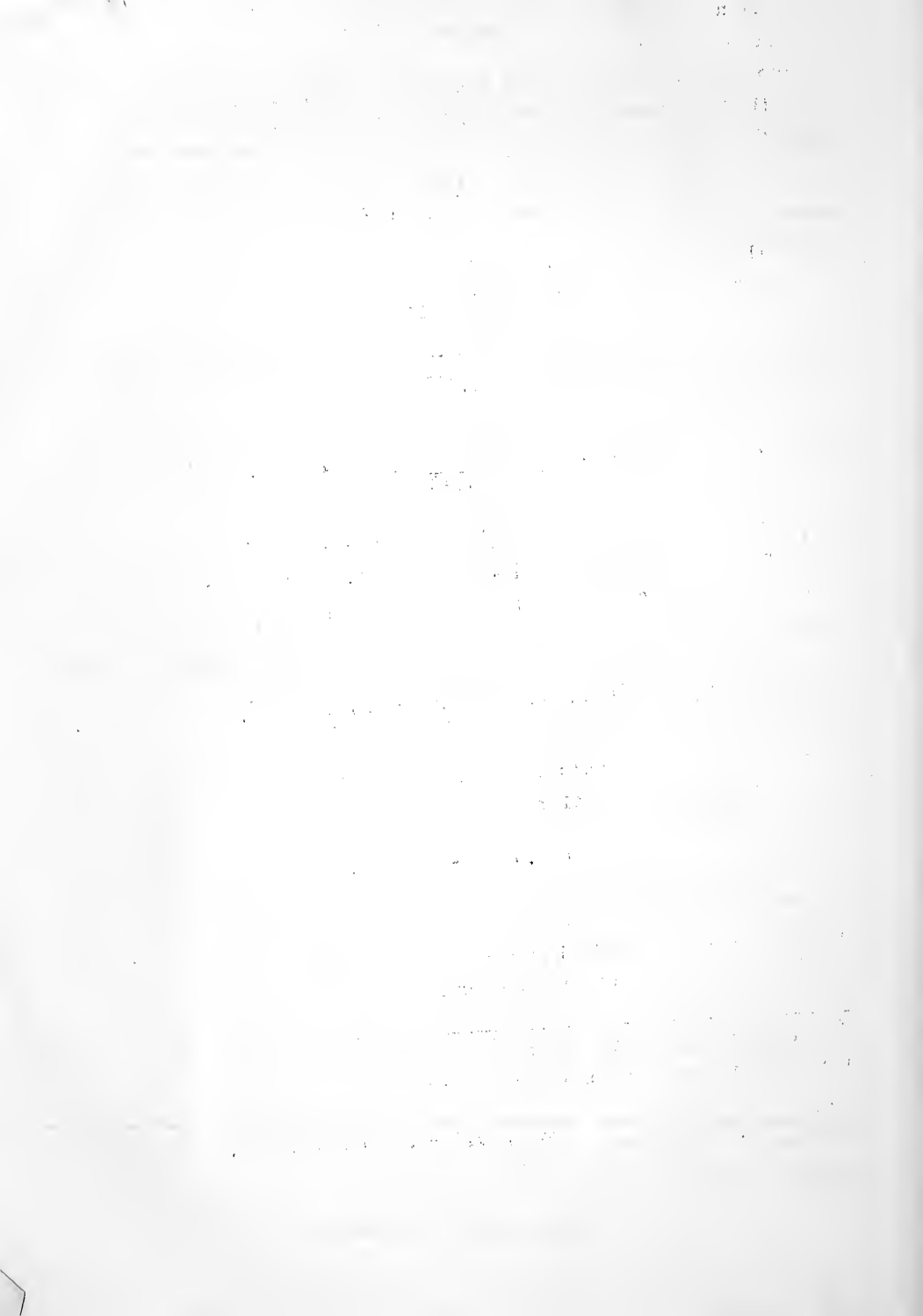
$$(22) \quad |Q[\omega_i, \omega_j] - \lambda H[\omega_i, \omega_j]| = 0 \quad ^* \quad .$$

It happens that the smallest root of this polynomial in  $\lambda$  is the desired approximation for the first eigen-value  $\lambda_1$  and that the subsequent roots are approximations to the higher eigen-

---

\* This notation denotes an  $n^{\text{th}}$  order determinant whose typical term in the  $i^{\text{th}}$  column and  $j^{\text{th}}$  row is

$$Q[\omega_i, \omega_j] - \lambda H[\omega_i, \omega_j] \quad .$$



values  $\lambda_2, \lambda_3, \dots, \lambda_n$ , although the latter approximations become worse and worse the farther on we go. If we desire not only good approximations to the lower eigen-values but to the higher ones as well, either the number  $n$  of coordinate functions must be taken very large or we must determine the first eigen-value and eigen-function with sufficient accuracy and proceed successively to the variational problem for the higher eigen-values. We must expect the approximations obtained to be greater than the actual values since the eigen-values are actually given by the minima of certain expressions.

We consider now an example to demonstrate the above method. We seek to determine the fundamental frequency (approximately) of a string with ends fixed at  $x = 0$  and  $x = 1$ . As coordinate functions we choose

$$\omega_i = x^{i+1}(1 - x) \quad , \quad (i = 0, 1, 2, \dots) \quad ,$$

which evidently satisfy the boundary conditions at  $x = 0$  and  $x = 1$ . It is clear that, no matter what admissible function we substitute in  $\bar{V}/\bar{T}$ , the result will not be less than  $\lambda_1$ . First, we try the simple function

$$\varphi_0 = x(1 - x) \quad .$$

Substituting into  $\bar{V}/\bar{T}$  we obtain

$$(22) \quad \frac{\bar{V}[\varphi_0, \varphi_0]}{\bar{T}[\varphi_0, \varphi_0]} = \frac{\int_0^1 \varphi_0'^2 dx}{\int_0^1 \varphi_0^2 dx} = \frac{\frac{1}{3}}{\frac{1}{30}} = 10 \quad .$$

If we try a slightly more complicated function

$$\varphi_1 = x(1 - x)(1 - ax) \quad ,$$

then

$$\frac{\bar{V}}{\bar{T}} = 14 \frac{5 - 5a + 3a^2}{7 - 7a + 2a^2}$$



which, for  $a = 0$ , has a minimum value of  $10$ . Thus we see that the addition of another coordinate function has produced no change whatsoever. Furthermore, if more coordinate functions are used, little or no change will be apparent, since the actual value of the first eigen-value for the string is  $\pi^2$ , which is very close to  $10$ . By employing polynomials of higher degree we may also obtain approximations for the higher eigen-values by using the theoretical method described above.

Exercises: 1. Using the function  $x(1 - x)(a + bx)$ , find approximately the first and second eigen-values for the string with fixed ends by determining the values of  $a$  and  $b$  which minimize  $\bar{V}/T$ .

2. Using the functions  $\varphi_0 = x(1 - x)y(1 - y)$  and  $\varphi_1 = x(1 - x)y(1 - y)(1 - ax - by)$ , find approximately the first eigen-value of the clamped plate occupying the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ .

3. Consider a string in the interval  $0 \leq x \leq 1$  which is fixed at  $x = 0$  and free to move at  $x = 1$ . Determine approximately the first eigen-value by using the function  $x(ax + b)$ . [Hint: Set up the determinant equation (22) and solve for  $\lambda$ ].

#### Part IV. Critical Remarks. Other Methods

The Rayleigh-Ritz method is really quite simple and straightforward. However, not only does it necessitate a great amount of computation but, also it possesses an inconvenience in that the accuracy of the result is not easily appraised. The reason for this lies in the fact that the approximations are always made from above and not from both sides of the actual value. Some recent progress, however, has been made in eliminating this short-coming by the development of new approximation methods likewise employing the concepts of the variational calculus.





One such method, and in some ways the most powerful and most promising, leads to equations of finite differences by replacing differential quotients by difference quotients and integrals by summations. Still another method, one that has not been used practically to any great extent, consists in replacing the differential quotients by certain integral expressions, e.g.,

$$\nabla^2 v \text{ by } \frac{1}{\pi h} \int_0^{2\pi} v(x + h \cos \theta, y + h \sin \theta) d\theta \quad .$$

This leads to a new approach and, in a certain sense, to problems similar to that of the random walk. Finally, there is a method which consists in replacing the differential equations by seemingly more complicated equations related to those of the theory of heat transfer, but which provide means for step-wise calculations.

In conclusion, we must apologize for the somewhat brief and sketchy character of our discussion of the Rayleigh-Ritz method and allied numerical methods in this course. A more detailed discussion, aimed at the solution of specific problems, is intended for a later offering.

QA935F41 BOSTON UNIVERSITY Scie  
Advanced methods in applied mathematics;



1 1719 00560 1200

**DO NOT REMOVE**

**CHARGE SLIP FROM THIS POCKET  
IF SLIP IS LOST PLEASE RETURN BOOK  
DIRECTLY TO A CIRCULATION STAFF MEMBER**



**BOSTON UNIVERSITY LIBRARIES**

771 Commonwealth Ave.  
Boston, Mass. 02215



